Preliminaries Given an equivalence relation $\sim$ on a set X, *the equivalence class containing* x is denoted $\overline{x}$ and defined by

$$\overline{x} := \{y \in X : y \sim x\}.$$

A subset S of X is *an equivalence class* if there exists $x \in X$ such that $S = \overline{x}$.

As an example, let $X = \mathbb{Z}$ and define $\sim$ by:

$m \sim n$ *if (and only if!)* $3|m - n$ *(that is, if* $m - n$ *is divisible by* 3).

Exercises: Verify that this $\sim$ is an equivalence relation. Then $\overline{0}$ is the set of all integers divisible by 3. What are $\overline{m}$, $1 \le m \le 6$? How many *different* $\overline{m}$'s are there? Each $\overline{m}$ is an equivalence class. Is $S := \{n \in \mathbb{Z} :$ There exists $j \in \mathbb{Z}$ such that $n = 3j + 7\}$ an equivalence class? If not, why not? If so, give an $m \in \mathbb{Z}$ such that $S = \overline{x}$. What about $S := \{n \in \mathbb{Z} :$ There exists $j \in \mathbb{Z}$ such that $n = 7j + 3\}$? This should give you a sense of what an equivalence class is.

In Ch 3 # 24c, the equivalence classes consist of all Cauchy sequences of points of X that are equivalent to each other, i.e., the ones that are equivalent to a particular one. Here is another

Exercise: *Prove that, if* $\{p_n\}$ *is Cauchy in* X *and* $\{p_{n_k}\}$ *is a subsequence, then* $\{p_{n_k}\} \sim \{p_n\}$.

This is an important Exercise, for we will use it a lot.

The definition of X^* is thus the collection of all equivalence classes of Cauchy sequences.

It might be handy to explicitly define another set: the set $\mathcal{X}$ of all Cauchy sequences of points of X. the equivalence relation $\sim$ is actually an equivalence relation on $\mathcal{X}$.

The last preliminary is the first of two Tricks we will use as part of the Warehouse Method. This Trick is a tool for constructing subsequences of convergent sequences or Cauchy sequences, that converge, or "cluster," as Cauchy sequences do, at a rate that we choose.

For convergent sequences: if $p_n \to p$, and $0 < \epsilon_k \to 0$ then for each k there exists a *first* $\hat{n}_k$ such that for all $n \ge \hat{n}_k$, $d(p_n, p) < \epsilon_k$. We define $n_k := k + \max_{1\le j\le k}\{\hat{n}_j\}$ (the Trick is adding the k). Then $n_k > \hat{n}_k$ (so that $d(p_{n_k}, p) < \epsilon_k$) and in addition $d(p_n, p) < \epsilon_k$ for all $n \ge n_k$. An important thing we need to know is that $\{p_{n_k}\}$ is a subsequence! We need to show that $n_{k+1} > n_k$: by definition

$$n_{k+1} = k + 1 + \max_{1\le j\le k+1}\{\hat{n}_j\} \ge 1 + k + \max_{1\le j\le k}\{\hat{n}_j\} = 1 + n_k, \quad \text{as desired.}$$

For Cauchy sequences: if $\{p_n\}$ is a Cauchy sequence and $0 < \epsilon_k \to 0$ then for each k there exists a *first* $\hat{n}_k$ such that for all $n \ge \hat{n}_k$, and for all $m \ge \hat{n}_k$, $d(p_n, p_m) < \epsilon_k$. As before we define $n_k := k + \max_{1\le j\le k}\{\hat{n}_j\}$. Then $n_k > \hat{n}_k$ (so that $d(p_{n_k}, p_n) < \epsilon_k$ if $n \ge n_k$), the n_k increase *strictly* and, in particular $d(p_{n_k}, p_{n_{k+1}}) < \epsilon_k$ for all k.

For example, if $\{a_n\}$ is a sequence of *numbers* that converges to limit a, the Trick gives us a subsequence $\{a_{n_k}\}$ such that $|a_{n_k} - a| < \epsilon_k$ *and* such that $|a_n - a| < \epsilon_k$ for all $n \ge n_k$. We will use this case later. There the numbers a_n will be complicated expressions, but they will still be numbers. The difficulty you'll have will be staying awake!

Filling the Warehouse We are given a Cauchy sequence of equivalence classes of Cauchy sequences: $\{P_n\}$, and this means that $\Delta(P_n, P_m) \to 0$ as $\min\{n, m\} \to \infty$. We apply the Cauchy sequence version of the Trick to $\{P_n\}$, using $\epsilon_k := 1/2^k$.

(1) We get n_k, strictly increasing, with $\Delta(P_{n_k}, P_{n_{k+1}}) < 1/2^k$ *and* with $\Delta(P_{n_k}, P_n) < 1/2^k$ for all $n \ge n_k$.

The equivalence classes P_{n_k} go into the Warehouse. We will in addition choose a sequence $\hat{p}_k := \{p_{k,j}\}_{j=1}^{\infty} \in P_{n_k}$, one for each k, and use the Trick to find a subsequence $\{p_{k,j_i}\}_{i=1}^{\infty}$ such that

(2) $d(p_{k,j_i}, p_{k,j_{i+1}}) < 1/2^i$ for all i, *and* such that $d(p_{k,j}, p_{k,j'}) < 1/2^i$ for all $j, j' \ge j_i$.

Here we have to remember that *this* j_i depends on k as well as i. Otherwise, we'd have to use really ugly notation, such as $j_{k,i}$. This subsequence $\{p_{k,j_i}\}$ is, by an Exercise called "important" above, in the same equivalence class as $\hat{p}_k$. We tend to use an element from an equivalence class as our "representative" member of the class. Sometimes we choose a "special" representative because of some useful property that it has, in addition to being a member of the equivalence class. To avoid messy subscript constructs, we will call the subsequence we constructed p'_k, with terms $p'_{k,i} := p_{k,j_i}$ and we will use p'_k, instead of $\hat{p}_k$, to "represent" P_{n_k}. In other words, we started with an arbitrary member of P_{n_k} and found a subsequence of it, using the Trick, and now we call that subsequence p'_k. Please note that $p'_k \in \mathcal{X}$, not X. The *terms* of p'_k, i.e. $p'_{k,i} = p_{k,j_i}$, are what belong to X.

Each Cauchy sequence p'_k is associated, in the Warehouse, with P_{n_k}. By definition,

$$\Delta(P_{n_k}, P_{n_{k+1}}) = \lim_{i\to\infty} d(p'_{k,i}, p'_{k+1,i}) \text{ and } \{d(p'_{k,i}, p'_{k+1,i})\}_{i=1}^{\infty} \text{ is a convergent sequence of } \textit{numbers.}$$

Once more we apply the Trick, using $\epsilon_\ell = 1/2^\ell$, and find a subsequence $\{d(p'_{k,I_\ell}, p'_{k+1,I_\ell})\}_{i=1}^{\infty}$ such that

(3) $$\left|d(p'_{k,I_\ell}, p'_{k+1,I_\ell}) - \Delta(P_{n_k}, P_{n_{k+1}})\right| < 1/2^\ell \text{ and } \left|d(p'_{k,i}, p'_{k+1,i}) - \Delta(P_{n_k}, P_{n_{k+1}})\right| < 1/2^\ell \text{ if } i \ge I_\ell.$$

The sequence $\{d(p'_{k,I_\ell}, p'_{k+1,I_\ell})\}_{\ell=1}^{\infty}$ goes into the Warehouse too, associated with P_{n_k}.

Setting up the solution: the IDEA We are given that $\{P_n\}$ is Cauchy in X^*. We need to show that the statement defining completeness is true. That is, we have to show that there exists $P \in X^*$ such that $\Delta(P_n, P) \to 0$ as $n \to \infty$. Once we have P we will show, using what we put in the Warehouse, that $\Delta(P_{n_k}, P) \to 0$ as $k \to \infty$. Then we can show that $P_n \to P$.

First we have to find a candidate for the limit, P. Here we have to use the IDEA that Cauchy sequences "cluster," and choose a term from each sequence p'_k that is a better and better approximation to what the sequence is "trying" to converge to, as k increases. Then we'll use the chosen terms to create a sequence p. We then have three tasks to perform:
(i) show that the sequence p we constructed is a Cauchy sequence (this puts p in $\mathcal{X}$),
(ii) show that, with $P := \overline{p}$, $\Delta(P_{n_k}, P) \to 0$ as $k \to \infty$ and
(iii) show that $\Delta(P_n, P) \to 0$ as $n \to \infty$. This task will be easy, compared to the others!

Looking at (2) and (3) suggests that we pick $i = k$ in (2) and pick $\ell = k$ in (3). Then we can define $L_k := \max\{j_k, I_k\}$ and try using p'_{k,L_k} to define p. That is, $p_k := p'_{k,L_k}$ for all k and $p = \{p_k\}$. What we did was choose a "far-out" term of each sequence p'_k as our k-th term of the newly made sequence p.

By the triangle inequality

$$d(p_k, p_{k+1}) = d(p'_{k,L_k}, p'_{k+1,L_{k+1}}) \le d(p'_{k,L_k}, p'_{k,L_{k+1}}) + d(p'_{k,L_{k+1}}, p'_{k+1,L_{k+1}}).$$

By (2) and our choice of $i = k$, $d(p'_{k,L_k}, p'_{k,L_{k+1}}) < 1/2^k$ since $L_k \ge j_k$. For the same reasons, but now from (3) and our choice $\ell = k$, $\left|d(p'_{k,L_{k+1}}, p'_{k+1,L_{k+1}}) - \Delta(P_{n_k}, P_{n_{k+1}})\right| < 1/2^k$. Therefore by (1)

$$d(p_k, p_{k+1}) = d(p'_{k,L_{k+1}}, p'_{k+1,L_{k+1}}) < 1/2^k + \Delta(P_{n_k}, P_{n_{k+1}}) \le 2/2^k.$$

Hence for all k,

(4) $$d(p_k, p_{k+1}) < 3/2^k.$$

The second Trick is to show the sequence p is Cauchy by adding the powers of $1/2$. So given $k < \ell$, we want a usable estimate for $d(p_k, p_\ell)$, so we use the triangle inequality as long as we have to to "get" from k to ℓ:

$$d(p_k, p_\ell) \le \sum_{j=k}^{\ell-1} d(p_j, p_{j+1}) < \sum_{j=k}^{\ell-1} 3/2^j < \sum_{j=k}^{\infty} 3/2^j = 6/2^k, \text{ OK because } \ell \text{ can be incredibly large.}$$

This shows that p is a Cauchy sequence! For, given $\epsilon > 0$ we find K so large that $6/2^K < \epsilon$. Then for all $k \ge K$ and $\ell \ge K$, $d(p_k, p_\ell) < \epsilon$. Thus $p \in \mathcal{X}$, so the equivalence class containing p, which we denote P, is a member of X^*. We have finished task (i).

We now turn to task (ii). We will refer to (2). By the definitions of $\Delta(P, Q)$ and $p'_{k,i}(= p_{k,j_i})$,

$$\Delta(P_{n_k}, P) = \lim_{i\to\infty} d(p'_{k,i}, p_i) = \lim_{i\to\infty} d(p'_{k,i}, p'_{i,L_i}) = \lim_{i\to\infty} d(p_{k,j_i}, p_{i,j_{L_i}}).$$

If $i \geq k$, then $L_i \geq i \geq k$, so we have, in (2) (second half), with $j := j_i \geq j_k$ and $j' := j_{L_i} \geq j_k$, that

$$d(p'_{k,i}, p_i) = d(p_{k,j_i}, p_{i,j_{L_i}}) = d(p_{k,j}, p_{i,j'}) < 1/2^i \leq 1/2^k.$$

But then $\Delta(P_{n_k}, P) = \lim_{i\to\infty} d(p'_{k,i}, p_i) \leq 1/2^k$ and therefore $P_{n_k} \to P$ in X^*. We have finished task (ii).

To complete task (iii), let $\epsilon > 0$ be given. We choose k so that $1/2^k < \epsilon/2$. We then suppose that $n \geq n_k$. Then there exists a unique $K \geq k$ such that $n_K \leq n < n_{K+1}$ and (now referring to (1))

$$\Delta(P_n, P) \leq \Delta(P_n, P_{n_K}) + \Delta(P_{n_K}, P) < 1/2^K + 1/2^K \leq 1/2^k + 1/2^k < \epsilon.$$

This completes task (iii) so the proof is complete.

Remark: The Tricks we used are based on some things we have done before. From time to time we constructed sequences converging to a limit point p of a set E in a metric space by choosing $p_n \in E$ such that $d(p_n, p) < 1/n$. In this way we constructed a sequence that converged to p at a *specified* rate (or maybe better!).

We also noted that if we know the partial sums s_n of a series with terms a_n, then $a_{n+1} = s_{n+1} - s_n$. We used this to show that if a series converges, its terms tend to zero. Then, by choosing a subsequence, we can arrange to have $|a_{n_k}| < 1/2^k$, and *these* terms tend to zero at a *specified* rate (or maybe better!). The choice of $1/2^k$ allows us estimate the tails $\sum_{k\geq K} |a_{n_k}|$ of our special "subseries." Usually, this does not help us much. But in the present problem, we were able to use these ideas many times! That's because we were not really seeking to show that a series converged, we were showing that a Cauchy sequence converges. So (another thing we have done) we broke the sequence down into blocks (from n_k to $n_{k+1} - 1$, essentially) that "clustered" together closer than $1/2^k$. Then it really did not matter much which term in there we used – the error we made by using one instead of another was at most $1/2^k$. And we could add in the subsequent errors and still get an error that was (possibly) bigger than $1/2^k$, to be sure, but only twice as big. That's because we used *specified* errors that were the terms of a geometric series.

In the sentence above that starts "But then $\Delta(P_{n_k}, P)$," we used the idea that if a sequence has a limit, then the sequence that starts with some very large subscript (we drop the first few million terms) has exactly the same limit. Therefore if we know that *eventually* the terms of our sequence are all bounded by a particular number, the limit will be bounded by that number too, though maybe with $\leq$ instead of $<$.