Complemented subspaces and quotients

Theorem 1.41, p 31, gives properties of quotient spaces.

Definition 4.20: *direct sum, complemented* (on p106).

Theorem 4.23, p 107, asserts that if T is in $\mathcal{K}(X)$, where X is a Banach space, and if $\lambda \neq 0$, then $S := T - \lambda I$ has closed range.

According to Lemma 4.21(a), p106, if M is finie-dimensional and X is locally convex, then M is complemented.

1 Lemma: *If a Banach space $X = M \oplus N$ then there exist unique continuous projections $P : X \to M$ and $Q : X \to N$ such that $\mathcal{R}(P) = M$, $\mathcal{R}(Q) = N$, $PQ = 0 = QP$ and $P + Q = I$.*

Proof: For every $x \in X$, $x = m + n$, where $m \in M$ and $n \in N$. If we also have $x \in X$, $x = m' + n'$, where $m' \in M$ and $n' \in N$, then $m - m' = n' - n$ so the vectors on both sides belong to $M \cap N = \{0\}$. Thus $m' = m$ and $n' = n$ so the representation $x = m + n$, where $m \in M$ and $n \in N$, is unique. We can thus define $Px := m$ and $Qx := n$. Then $P + Q = I$. Each of P and Q is linear because of the unique representation. Moreover, each of P and Q maps onto M and N respectively, because each is the identity on M and N respectively. The operators P and Q are uniquely determined by the unique representation $x = m+n$. The condition $M \cap N = \{0\}$ shows that $PQ = 0 = QP$. To show continuity we apply the Closed Graph Theorem.

Let $x_n \to x$ and $Px_n \to y$. Then, since M is closed $y \in M$. Thus $Py = y$. We also have $x_n - Px_n \in N$ and $x_n - Px_n \to x - y$, so $x - y \in N$. But then $P(x - y) = 0 = Px - Py = Px - y$, so $Px = y$, the graph of P is closed, and so P is continuous, as is $Q = I - P$.

Remark: In $\mathbb{R}^2$ with the usual Euclidean norm, the subspace M, given by the real axis, and the subspace N given by the line thru the origin with slope 1 have projections $P = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. The norms are not 1. In fact, $\|P\| = \sqrt{2} = \|Q\|$. We note that 1 is the least possible norm that a non-zero projection operator in $\mathcal{B}(X)$ can have.

2 Lemma: *If $X = M \oplus N$ then $N \cong X/M$*

Proof: We will use the projections P and Q and the "usual" quotient map $\tau : X \to X/M$, given by $x \overset{\tau}{\mapsto} x + M$. We define $\sigma : N \to X/M$ by $\sigma n := n + M$. Then $\sigma n = \sigma n'$ means $n + M = n' + M$, or that $n - n' \in M$, so $n' = n$. To show that σ is onto, we let $x \in X$ be arbitrary. Then $\tau x = x + M = m + n + M = n + M = \sigma Q x$, so σ is onto.

To show continuity, we have

$$\|\sigma n\| = \inf_{m \in M} \|n - m\| \leq \|n\|.$$

We apply Corollary 2.12(b), p 49, to complete the proof.

3 Lemma: *If $\{y_1, \ldots, y_n\}$ is a linearly independent finite set in a TVS X such that X^* separates points, then there exists a set $\{y_1^*, \ldots, y_n^*\} \subseteq X^*$ such that $\langle y_i, y_j^* \rangle = \delta_{ij}$ for all i and j in $\{1, \ldots, n\}$.*

Proof: Let $\Lambda : X^* \to \mathbb{C}^n$ be given by $\Lambda x^* := (x^*(y_1), \ldots, x^*(y_n))$. We can show that Λ is wk*-continuous and onto. This will give us the result. Given $\epsilon > 0$, if $y^* \in V := \{x^* \in X^* : |\langle y_k, x^* \rangle| < \epsilon/\sqrt{n},\ 1 \leq k \leq n\}$ then $\|\Lambda x^*\|_{\mathbb{C}^n} < \sqrt{n \frac{\epsilon^2}{n}} = \epsilon$. Since $\mathcal{R}(\Lambda)$ is a finite dimensional subspace, it is closed in $\mathbb{C}^n$. We have

$$\mathcal{R}(\Lambda)^\perp = \left\{ z \in \mathbb{C}^n : \sum_{k=1}^n z_k \langle y_k, x^* \rangle = 0 \text{ for all } x^* \in X^* \right\}.$$

But if, for all $x^* \in X^*$, $0 = \sum_{k=1}^n z_k \langle y_k, x^* \rangle = \left\langle \sum_{k=1}^n z_k\, y_k\,,\, x^* \right\rangle = x^* \left(\sum_{k=1}^n z_k\, y_k \right)$, then $\sum_{k=1}^n z_k\, y_k = 0$ because X^* separates points. Since $\{y_1, \ldots, y_n\}$ is linearly independent, $z_k = 0$ for every k. Hence $\mathcal{R}(\Lambda)^\perp = \{0\}$ so $\mathcal{R}(\Lambda) = \mathbb{C}^n$. But then for each basis element e_k of $\mathbb{C}^n$ there exists $y_k^* \in X^*$ such that $\Lambda y_k^* = e_k$, as desired.

4 Lemma: *Let X be a TVS, and $\{x_1^*, \ldots, x_m^*\} \subseteq X^*$ a linearly independent finite set. Then there exists a set $\{x_1, \ldots, x_m\} \subseteq X$ such that $\langle x_i, x_j^* \rangle = \delta_{ij}$ for all i and j in $\{1, \ldots, m\}$.*

Proof: Let us put the weak-star topology on X^*, i.e. the X-topology. With this topology X^* is locally convex. Let us put $M_* = \operatorname{span}\{x_1^*, \ldots, x_m^*\} \subseteq X^*$. When we write $x^* \in M_*$ as $x^* = \sum_{j=1}^m \alpha_j x_j^*$, the uniqueness of the coefficients implies that each is a linear functional on M_*. Since M_* has finite dimension, each $\alpha_j = \alpha_j(x^*)$ is continuous on M_*. We can now extend each to a linear functional on X^*, continuous with respect to the X-topology. Since these linear functionals are precisely those determined by the elements of X, we have the existence of $\{x_1, \ldots, x_m\} \subseteq X$ such that for each j, $1 \le j \le m$, we have: $x^* \mapsto \langle x_j, x^* \rangle$ is an extension of α_j. This means that $\langle x_i, x_j^* \rangle = \delta_{ij}$, as desired (because $\alpha_j(x_i^*) = \delta_{ji}$).

5 Theorem: *If X is a Banach space and $T \in \mathcal{K}(X)$ then for every non-zero scalar λ the four spaces*

$$\mathcal{N}(T - \lambda I),\ X/\mathcal{R}(T - \lambda I),\ \mathcal{N}(T^* - \lambda I) \text{ and } X^*/\mathcal{R}(T^* - \lambda I)$$

have the same, finite, dimension.

We know that $m := \dim \mathcal{N}(T^* - \lambda I) < \infty$ (Theorems 4.18d, p104 and 4.19, p 105). As Part One of the proof we will show that, with $S := T - \lambda I$, $\mathcal{N}(S^*) \cong X/\mathcal{R}(S)$.

We recall from the theory of systems of linear equations that for a $p \times q$ matrix A, $\mathcal{R}(A)^\perp = \mathcal{N}(A^T)$. Therefore $p = \dim \mathcal{R}(A) + \dim \mathcal{N}(A^T)$. Let us therefore try to show that there exists a subspace M of X, isomorphic to $\mathcal{N}(S^*)$, such that

6 $$X = \mathcal{R}(S) \oplus M.$$

Then by Lemma 2, $X/\mathcal{R}(S) \cong M$ $(\cong \mathcal{N}(S^*))$ and we are done. It remains to prove Equation 6. We use Lemma 4 to find a *possible* M, starting with a basis $\{x_1^*, \ldots, x_m^*\}$ of $\mathcal{N}(S^*)$. We obtain a (linearly independent!) set $\{x_1, \ldots, x_m\} \subseteq X$ such that $\langle x_i, x_j^* \rangle = \delta_{ij}$ for all i and j in $\{1, \ldots, m\}$, and we define $M := \operatorname{span}\{x_1, \ldots, x_m\}$.

We will digress briefly. Let us write

$$Qx := \sum_{j=1}^m \langle x, x_j^* \rangle\, x_j.$$

Then Q is continuous and $\mathcal{R}(Q) = M$ because $Qx \in M$ for all x, and because $Qx_j = x_j$ for each j, $1 \le j \le m$. Moreover, $Q^2 = Q$ so Q is a continuous projection with range M.

Let us find the corresponding formula for Q^*. We have

$$\langle x, Q^* x^* \rangle = \langle Qx, x^* \rangle = \left\langle \sum_{j=1}^m \langle x, x_j^* \rangle\, x_j,\, x^* \right\rangle = \sum_{j=1}^m \langle x, x_j^* \rangle \langle x_j, x^* \rangle = \left\langle x,\, \sum_{j=1}^m \langle x_j, x^* \rangle\, x_j^* \right\rangle$$

Thus

$$Q^* x^* = \sum_{j=1}^m \langle x_j, x^* \rangle\, x_j^*.$$

Thus Q^* is a continuous projection with range $\mathcal{N}(S^*)$.

We return to the proof of Equation 6. Suppose that for some u and some x, $Su = Qx$. Then

$$Su = Qx = \sum_{j=1}^m \langle x, x_j^* \rangle\, x_j.$$

Then for each k, $1 \le k \le m$,

$$\langle Su, x_k^* \rangle = \langle u, S^* x_k^* \rangle = 0, \quad \text{since } x_k^* \in \mathcal{N}(S^*), \quad \text{and so}$$

$$\langle Su, x_k^* \rangle = \left\langle \sum_{j=1}^m \langle x, x_j^* \rangle\, x_j,\, x_k^* \right\rangle = \langle x, x_k^* \rangle = 0.$$

Hence $Qx = 0 = Su$, and thus $\mathcal{R}(S) \cap M = \{0\}$.

To finish the proof of Equation 6, we need to show that $\mathcal{R}(S) + M = X$. It will be enough to show that for all $x \in X$, $x - Qx \in \mathcal{R}(S)$.

We will show that $x - Qx \in {}^{\perp}\mathcal{N}(S^*)$. For all $x^* \in \mathcal{N}(S^*)$,

$$\langle x - Qx,\, x^*\rangle = \langle x,\, x^*\rangle - \langle Qx,\, x^*\rangle = \langle x,\, x^*\rangle - \langle x,\, Q^*x^*\rangle = \langle x,\, x^*\rangle - \langle x,\, x^*\rangle = 0,$$

so indeed we have $x - Qx \in {}^{\perp}\mathcal{N}(S^*) = {}^{\perp}(\mathcal{R}(S))^{\perp} = \overline{\mathcal{R}(S)} = \mathcal{R}(S)$ by Theorems 4.7, p 96, 4.12, p 99 and 4.23, on p 107. This completes the proof of Equation 6.

We can use essentially the same argument to prove that $\mathcal{N}(S) \cong X^*/\mathcal{R}(S^*)$. There are some minor differences. The main idea is that X^* is, after all, a Banach space. The details are in Appendix 1.

We have shown that $\mathcal{N}(S^*) \cong X/\mathcal{R}(S)$ and that $X^*/\mathcal{R}(S^*) \cong \mathcal{N}(S)$. To complete the proof of Theorem 5 it is thus enough to prove that $\mathcal{N}(S^*) \cong \mathcal{N}(S)$.

We begin this by showing that $\mathcal{N}(S^*) = \{0\}$ implies that $\mathcal{N}(S) = \{0\}$.

If we are given that $\mathcal{N}(S^*) = \{0\}$, then $X = {}^{\perp}\mathcal{N}(S^*) = \mathcal{R}(S)$, as we saw at the end of the proof of Equation 6. Hence S is onto. To get a contradiction let us assume there is $x_1 \neq 0$ such that $Sx_1 = 0$.

An argument for showing that an operator is NOT a compact operator

We will construct a sequence $\{z_n\}$ with $\|z_n\| = 1$ such that $\{Tz_n\}$ contains no convergent subsequence.

By induction we select a sequence $\{x_n\}$ such that $Sx_{n+1} = x_n$. Since $S = T - \lambda I$, we have $Tx_{n+1} = \lambda x_{n+1} + x_n$. Therefore

$$S\{x_1,\, \ldots,\, x_{n+1}\} \subseteq \{x_1,\, \ldots,\, x_n\} \quad \text{and} \quad T\{x_1,\, \ldots,\, x_{n+1}\} \subseteq \operatorname{span}\{x_1,\, \ldots,\, x_{n+1}\}.$$

Let us define subspaces $M_n := \operatorname{span}\{x_1,\, \ldots,\, x_n\}$. Then we have

7 $$SM_{n+1} \subseteq M_n \quad \text{and} \quad TM_n \subseteq M_n.$$

Indeed, we note for immediate use that $S^m x_j = 0$ if $m \geq j$ and $S^m x_j \neq 0$ if $0 \leq m < j$. This all follows from the identity $S^m x_j = x_{\max\{0,\, j-m\}}$, where $x_0 := 0$ is not a part of the sequence $\{x_n\}$.

Claim: *The sequence $\{x_n\}$ is linearly independent.*

Proof: Suppose F is a finite set of integers such that $\sum_{k\in F} \alpha_k x_k = 0$, but no $\alpha_k = 0$. Then if $M = \max F$, $0 = S^{M-1}\sum_{k\in F}\alpha_k x_k = S^{M-1}\alpha_M x_M = \alpha_M x_1 \neq 0$, a contradiction.

It follows now that $M_n \subset M_{n+1}$ but $M_n \neq M_{n+1}$. We will use a geometric Lemma.

8 Lemma: *If X is a normed space and M is a closed subspace that is not all of X then there exists $z \in X$ such that $\|z\| = 1$ and $\|z - m\| > 1/2$ for all $m \in M$.*

Proof: Suppose $\xi \notin M$. Then $0 < \delta = \inf_{m\in M}\|\xi - m\|$. We choose $\mu \in M$ such that $0 < \delta \leq \|\xi - \mu\| < 2\delta$. Then $z := \dfrac{\xi - \mu}{\|\xi - \mu\|}$ will do, for whenever $m \in M$,

$$\|z - m\| = \left\|\frac{\xi - \mu}{\|\xi - \mu\|} - m\right\| = \frac{\|\xi - (\mu + \|\xi - \mu\|\, m)\|}{\|\xi - \mu\|} \geq \frac{\delta}{\|\xi - \mu\|} > \frac{\delta}{2\delta} = 1/2, \quad \text{as desired.}$$

Each of our subspaces M_n is closed, so for $n > 1$ we can select $z_n \in M_n$ such that $\|z_n\| = 1$ and $\|z_n - m\| > 1/2$ for all $m \in M_{n-1}$. The sequence $\{z_n\}$ is the one we want. Let us suppose that $1 < k < n$. Then since $Tx = Sx + \lambda x$, and since $Sz_n \in M_{n-1}$ and $Tz_k \in M_k \subseteq M_{n-1}$,

$$\|Tz_n - Tz_k\| = \|\lambda z_n - (-Sz_n + Tz_k)\| = |\lambda|\left\|z_n - \frac{-Sz_n + Tz_k}{\lambda}\right\| > |\lambda|/2.$$

Hence no subsequence of $\{Tz_n\}$ can converge. This contradicts the compactness of T. Thus $\mathcal{N}(S) = \{0\}$, as desired.

An observation about the argument

The properties of T that we used are: T has a non-zero eigenvalue λ such that $T - \lambda I$ is onto. The argument thus shows that no linear operator $T : X \to X$ with these properties can be a compact operator on X.

Having shown that $\mathcal{N}(S^*) = \{0\}$ implies that $\mathcal{N}(S) = \{0\}$, we next show that $\mathcal{N}(S) = \{0\}$ implies that $\mathcal{N}(S^*) = \{0\}$. We can use almost exactly the same argument. We just have to refer to the end of the proof of Equation A1 in Appendix 1 to see that $\mathcal{N}(S) = \{0\} \Rightarrow \mathcal{R}(S^*) = X^*$.

So far we know $\mathcal{N}(S^*) = \{0\} \iff \mathcal{N}(S) = \{0\}$. We want to know $\dim \mathcal{N}(S^*) = \dim \mathcal{N}(S)$. To accomplish this we assume that $m := \dim \mathcal{N}(S^*) > 0$ and $n := \dim \mathcal{N}(S) > 0$. We will construct a rank-one (hence compact) operator C such that $\dim \mathcal{N}(S - C) = n - 1$ and $\dim \mathcal{N}(S^* - C^*) = m - 1$. We can repeat the reductions in dimension until one of them is zero. At that point we know the other dimension is zero as well, so we must have had $m = n$ to begin with.

We will use the bases for our null spaces and the corresponding related spaces M and M_* that came from Lemmas 3 and 4, and that were used in Part One of the proof and in Appendix 1.

The idea is to *mix* the projections we have constructed: we define

$$Cx := \langle x,\, y_1^* \rangle\, x_1, \quad \text{so that} \quad C^* x^* := \langle x_1,\, x^* \rangle\, y_1^*.$$

If now $x \neq 0$ and $(S - C)x = 0$ this means $Sx = Cx = \langle x,\, y_1^* \rangle\, x_1 \in M$, but $\mathcal{R}(S) \cap M = \{0\}$, so $\langle x,\, y_1^* \rangle = 0$ and so $Sx = 0 = Cx$. Thus $x \in \mathcal{N}(S)$. Hence $x = \sum_{j=1}^{n} \langle x,\, y_j^* \rangle\, y_j = \sum_{j>1} \langle x,\, y_j^* \rangle\, y_j$. Thus $\dim \mathcal{N}(S - C) \leq n - 1$. On the other hand, if $\xi = \sum_{j>1} \alpha_j y_j$, then $S\xi = 0 = C\xi$, so $\xi \in \mathcal{N}(S - C)$, hence $\dim \mathcal{N}(S - C) \geq n - 1$. The details that show $\dim \mathcal{N}(S^* - C^*) = m - 1$ occupy Appendix 2.

This completes the proof of Theorem 5.

Theorem 5 gives us this Corollary: Every non-zero element of the spectrum of a compact operator is an eigenvalue.

9 Corollary: *If $0 \neq \lambda \in \sigma(T)$ then λ is an eigenvalue of both T and T^*.*

Proof: If $\dim \mathcal{N}(S) = 0$ then $\dim \mathcal{N}(S^*) = 0$ which (as we have seen) implies that S is onto, and hence $T - \lambda I$ is invertible, so $\lambda \notin \sigma(T)$. Thus $\lambda \in \sigma(T)$ implies $\dim \mathcal{N}(S) = \dim \mathcal{N}(S^*) > 0$, so λ is an eigenvalue of both T and T^*.

To introduce the last result we will prove, let us note a general property of operators $T \in \mathcal{B}(X)$.

10 Theorem: *If $T \in \mathcal{B}(X)$ then $\sigma(T)$ is compact, and is contained in the disc of radius $\inf_{n \geq 1} \|T^n\|^{1/n}$ and center zero. In particular, if λ is in the spectrum of T then $|\lambda| \leq \|T\|$.*

Proof: First let us prove an important theorem.

11 Theorem: *If $T \in \mathcal{B}(X)$ is invertible, then for all $U \in \mathcal{B}(X)$ such that $\|T - U\| < 1/\left\|T^{-1}\right\|$, U is invertible. Thus the set of invertible elements of $\mathcal{B}(X)$ is open.*

Proof: We will use the convergence of geometric series. We have

$$U = T + (U - T) = T(I - T^{-1}(T - U)). \text{ Let us put } H := T^{-1}(T - U). \text{ By hypothesis, } \|H\| < 1.$$

Since $\|H^n\| \leq \|H\|^n$, $\sum_{n=0}^{\infty} H^n$ converges in $\mathcal{B}(X)$. Since $(I - H) \sum_{n=0}^{N} H^n = I - H^{N+1} = \sum_{n=0}^{N} H^n (I - H)$, we have $V := \sum_{n=0}^{\infty} H^n = (I - H)^{-1}$ and $I - H = V^{-1}$ as well. Let us show that VT^{-1} is the inverse of U. This simply means we have to show that $(VT^{-1})U = I = U(VT^{-1})$. Now $T^{-1}U = I - H$ by definition. By the associativity of operator composition, $(VT^{-1})U = V(I - H) = I$.

The other direction may be a little tricky:

$$U(VT^{-1}) = TT^{-1}UVT^{-1} = TV^{-1}VT^{-1}UVT^{-1} = TV^{-1}(VT^{-1}U)VT^{-1} = TV^{-1}IVT^{-1} = I.$$

Hence U is invertible. In the proof we proved the following Corollary.

12 Corollary: *If* $T \in \mathcal{B}(X)$ *and* $\|T\| < 1$ *then* $\|I - T\|$ *is invertible and* $\left\|(I-T)^{-1}\right\| \le 1/(1 - \|T\|)$.

To return to the proof of Theorem 10, we note that if $\lambda \notin \sigma(T)$ then $S_\lambda := T - \lambda I$ is invertible. We then have $\|S_\lambda - S_\mu\| = |\lambda - \mu| < 1/\left\|S_\lambda^{-1}\right\|$ if $|\lambda - \mu|$ is small enough. Thus $\{\lambda \in \mathbb{C} : S_\lambda \text{is invertible }\}$ is open. Thus $\sigma(T)$ is closed.

To show that $\sigma(T)$ is bounded we use Corollary 12. If $|\lambda| > \|T\|$ then $I - (1/\lambda)T$ is invertible, so $T - \lambda I = (-\lambda)(I - (1/\lambda)T)$ is invertible as well. In particular, $\sigma(T)$ is contained in the closed disc, center 0, of radius $\|T\|$.

If $T - \lambda I$ is not invertible, then neither is $T^n - \lambda^n I$. Thus $|\lambda|^n \le \|T^n\|$. This essentially completes the proof of Theorem 10.

13 Theorem: *If* $T \in \mathcal{K}(X)$ *then* $\sigma(T)$ *is countable and has at most one limit point, namely zero (which is in the spectrum if* $\dim X = \infty$*).*

Proof: That 0 is in $\sigma(T)$ if $\dim X = \infty$ is shown in Theorem 4.18(e), p 104. Suppose, on the contrary, that $\{\lambda_n\}$ is a sequence of distinct eigenvalues that converges to $\lambda \ne 0$. Then $|\lambda_n| > |\lambda|/2$ for all n sufficiently large. Without loss of generality we assume that $|\lambda_n| > |\lambda|/2$ for all n. We will again use the "argument for showing that an operator is NOT a compact operator."

We select an eigenvector x_n belonging to λ_n, for each n. Since the eigenvalues are distinct the nested spaces

$$M_n := \operatorname{span}\{x_1, \ldots, x_n\}$$

have dimension n so the M_n are strictly increasing. Since each M_n has a basis of eigenvectors, $TM_n \subseteq M_n$. On the other hand, if $n > 1$ and $x \in M_n$, then $x = \sum_{k=1}^n \alpha_k x_k$ so $S_{\lambda_n} x = \sum_{k=1}^n \alpha_k(\lambda_k - \lambda_n)x_k \in M_{n-1}$ since the last term in the sum is zero. This shows us that $S_{\lambda_n} M_n \subseteq M_{n-1}$. By Lemma 8 we choose for each $n > 1$ a $z_n \in M_n$ such that $\|z_n\| = 1$ and $\|z_n - m\| > 1/2$ for all $m \in M_{n-1}$.

Now we suppose that $n > k > 1$ and examine the norm of $Tz_n - Tz_k$. To begin we recall that $T = S_{\lambda_n} + \lambda_n I$, so

$$Tz_n - Tz_k = \lambda_n z_n - (-S_{\lambda_n} z_n + Tz_k) = \lambda_n \left(z_n - \frac{-S_{\lambda_n} z_n + Tz_k}{\lambda_n} \right), \quad \text{and} \quad \frac{-S_{\lambda_n} z_n + Tz_k}{\lambda_n} \in M_{n-1},$$

so that

$$\|Tz_n - Tz_k\| = |\lambda_n| \left\| z_n - \frac{-S_{\lambda_n} z_n + Tz_k}{\lambda_n} \right\| > |\lambda_n|/2 > |\lambda|/4.$$

Thus T cannot be a compact operator, which gives us the contradiction we want. Hence the only possible limit point of the set of non-zero eigenvalues is zero. So, every non-zero eigenvalue has a neighborhood containing no others. Thus the set of non-zero eigenvalues is countable. This completes our alternate proof of Theorem 4.25, p 108.

Another observation about the argument

Here we assumed that T had a sequence of distinct eigenvalues with a limit point different from zero. Thus no such operator can be compact. Indeed, if $\lambda \ne \lambda_n \to \lambda \ne 0$ and $\lambda_n \in \sigma(T)$ for each n, then T cannot be compact.

Appendix 1: Details of the proof that $\mathcal{N}(S) \cong X^*/\mathcal{R}(S^*)$.

We want to show that there exists a subspace M_* of X^*, isomorphic to $\mathcal{N}(S)$, such that

A1 $$X^* = \mathcal{R}(S^*) \oplus M_*.$$

As before, by Lemma 2, $X^*/\mathcal{R}(S^*) \cong M_*$ $(\cong \mathcal{N}(S))$ and we are done. To prove Equation A1 we use Lemma 3 to find a *possible* M_*, starting with a basis $\{y_1, \ldots, y_n\}$ of $\mathcal{N}(S)$. We obtain a (linearly independent!) set $\{y_1^*, \ldots, y_n^*\} \subseteq X^*$ such that $\langle y_i, y_j^* \rangle = \delta_{ij}$ for all i and j in $\{1, \ldots, n\}$, and we define $M_* := \operatorname{span}\{y_1^*, \ldots, y_n^*\}$.

We will digress briefly. Let us write

$$Q_* x^* := \sum_{k=1}^{n} \langle y_k, \, x^* \rangle \, y_k^*.$$

Then Q_* is continuous and $\mathcal{R}(Q_*) = M_*$ because $Q_* x^* \in M_*$ for all x^*, and because $Q_* y_k^* = y_k^*$ for each k, $1 \le k \le n$. Moreover, $Q_*^2 = Q_*$ so Q_* is a continuous projection with range M_*.

Let us find a corresponding formula for a "predual" operator ${}^*Q$. We have

$$\langle x, \, Q_* x^* \rangle = \left\langle x, \, \sum_{k=1}^{n} \langle y_k, \, x^* \rangle \, y_k^* \right\rangle = \sum_{k=1}^{n} \langle x, \, y_k^* \rangle \, \langle y_k, \, x^* \rangle = \left\langle \sum_{k=1}^{n} \langle x, \, y_k^* \rangle \, y_k, \, x^* \right\rangle$$

Thus

$${}^*Q x = \sum_{k=1}^{n} \langle x, \, y_k^* \rangle \, y_k.$$

We have shown that ${}^*Q$ is a continuous projection with range $\mathcal{N}(S)$. But now we have $({}^*Q)^* = Q_*$. Let us therefore revert to the earlier notation, by replacing ${}^*Q$ by Q and by replacing Q_* by Q^*, with the understanding that Q is not the same operator we had before, related to Equation 6!

We return to the proof of Equation A1. Suppose that for some u^* and some x^*, $S^* u^* = Q^* x^*$. Then

$$S^* u^* = Q^* x^* = \sum_{k=1}^{n} \langle y_k, \, x^* \rangle \, y_k^*.$$

Then for each j, $1 \le j \le n$,

$$\langle y_j, \, Q^* x^* \rangle = \langle y_j, \, S^* u^* \rangle = \langle S y_j, \, u^* \rangle = 0, \quad \text{since } \; y_j \in \mathcal{N}(S), \quad \text{and so}$$

$$\langle y_j, \, Q^* x^* \rangle = \left\langle y_j, \, \sum_{k=1}^{n} \langle y_k, \, x^* \rangle \, y_k^* \right\rangle = \langle y_j, \, x^* \rangle = 0.$$

Hence $Q^* x^* = 0 = S^* u^*$, and thus $\mathcal{R}(S^*) \cap M_* = \{0\}$.

To finish the proof of Equation A1, we need to show that $\mathcal{R}(S^*) + M_* = X^*$. It will be enough to show that for all $x^* \in X^*$, $x^* - Q^* x^* \in \mathcal{R}(S^*)$.

We will show that $x^* - Q^* x^* \in \mathcal{N}(S)^\perp$. For all $x \in \mathcal{N}(S)$, we know that $Qx = x$, so

$$\langle x, \, x^* - Q^* x^* \rangle = \langle x, \, x^* \rangle - \langle x, \, Q^* x^* \rangle = \langle x, \, x^* \rangle - \langle Qx, \, x^* \rangle = \langle x, \, x^* \rangle - \langle x, \, x^* \rangle = 0.$$

Thus we have $x^* - Q^* x^* \in \mathcal{N}(S)^\perp = ({}^\perp \mathcal{R}(S^*))^\perp = \overline{\mathcal{R}(S^*)}^{w^*} = \mathcal{R}(S^*)$ by Theorems 4.7, p96, 4.12, p 99, 4.14 on page 101 and 4.23, p 107. This completes the proof of Equation A1.

Appendix 2: Details of the proof that $\dim \mathcal{N}(S^* - C^*) = m - 1$.

We recall definitions:

$$Cx := \langle x, \, y_1^* \rangle \, x_1, \quad \text{so that } \; C^* x^* := \langle x_1, \, x^* \rangle \, y_1^*.$$

If now $x^* \neq 0$ and $(S^* - C^*) x^* = 0$ this means $S^* x^* = C^* x^* = \langle x_1, \, x^* \rangle \, y_1^* \in M_*$, but $\mathcal{R}(S^*) \cap M_* = \{0\}$, so $\langle x_1, \, x^* \rangle \, y_1^* = 0$ and so $S^* x^* = 0 = C^* x^*$. Thus $x^* \in \mathcal{N}(S^*)$. Hence $x^* = \sum_{j=1}^{m} \langle x_j, \, x^* \rangle \, x_j^* = \sum_{j>1} \langle x_j, \, x^* \rangle \, x_j^*$. Thus $\dim \mathcal{N}(S^* - C^*) \le m - 1$. On the other hand, if $\xi^* = \sum_{j>1} \alpha_j x_j^*$, then $S^* \xi^* = 0 = C^* \xi^*$, so $\xi^* \in \mathcal{N}(S^* - C^*)$, hence $\dim \mathcal{N}(S^* - C^*) \ge m - 1$.