Theorem *If* $A \subseteq \mathbb{R}^\ell$ *and* $B \subseteq \mathbb{R}^m$ *are compact, Then* $A \times B \subseteq \mathbb{R}^{\ell+m}$ *is compact.*

Proof: Suppose that $\mathcal{U}$ is an open cover of $A \times B$. We will replace $\mathcal{U}$ with $\mathcal{C}$, where

$$\mathcal{C} := \{B_r\big((x,\,y)\big) : r > 0,\ (x,\,y) \in A \times B,\ B_r\big((x,\,y)\big) \subseteq U \ \text{ for some } \ U \in \mathcal{U}\}.$$

Since every point of $A \times B$ belongs to a ball contained in some U belonging to $\mathcal{U}$, $\mathcal{C}$ is an open cover of $A \times B$. Let us note that $\mathcal{C}$ contains all the balls that it possibly can, and for *every* point $(x,\,y) \in A \times B$ there are infinitely many $B_r\big((x,\,y)\big)$ in $\mathcal{C}$.

We will need to get open covers of A and of B to take advantage of their compactness.

A technical digression It will be handy to use the two inclusions

$$B_{r/\sqrt{2}}(x) \times B_{r/\sqrt{2}}(y) \subseteq B_r\big((x,\,y)\big) \subseteq B_r(x) \times B_r(y).$$

The first inclusion *could* be written $B_{r\cos\theta}(x) \times B_{r\sin\theta}(y) \subseteq B_r\big((x,\,y)\big)$, with $0 < \theta < \pi/2$, for if $x' \in B_{r\cos\theta}(x)$ and $y' \in B_{r\sin\theta}(y)$ then $|x'-x|^2 + |y'-y|^2 < r^2\cos^2\theta + r^2\sin^2\theta = r^2$. But this would require that we had already proved something about sine and cosine, and we have not. So we will simply use a fact we *have* proved, namely that $1/\sqrt{2}$, when squared and doubled, gives 1. The second inclusion is immediate, for $d((x',\,y'),\,(x,\,y))^2 = |x'-x|^2 + |y'-y|^2 < r^2$ implies that each of $|x'-x|$ and $|y'-y|$ is less than r. **end of digression**

Now let us fix a point x_o of A. We will define an open cover of B as follows:

$$\mathcal{B}_{x_o} := \{B_{r/\sqrt{2}}(y) : y \in B \ \text{ and } \ B_r\big((x_o,\,y)\big) \in \mathcal{C}\}.$$

By the compactness of B this open cover $\mathcal{B}_{x_o}$ of B has a finite subcover

$$\Big\{B_{r_i(x_o)/\sqrt{2}}\big(y_i(x_o)\big) : i = 1,\,\ldots,\,N_{x_o}\Big\}$$

for some positive integer N_{x_o}, which, with the open cover, depends on x_o. At the moment we need a Lebesgue number for it. We know from the proof of the Theorem on compactness and limit points that there is a number $\epsilon > 0$ such that for all $y \in B$, $B_\epsilon(y)$ is contained in one of the sets $B_{r_i(x_o)/\sqrt{2}}\big(y_i(x_o)\big)$.

To prepare for the next step let us define

$$\epsilon(x_o) := \min\{\epsilon,\, r_1/\sqrt{2},\,\ldots,\,r_N/\sqrt{2}\ \} \ \text{ and } \ \mathcal{F}_{x_o} := \{B_{r_i(x_o)/\sqrt{2}}\big(y_i(x_o)\big) : i = 1,\,\ldots,\,N_{x_o}\}.$$

All the quantities in these definitions depend on x_o.

We have some more "big" open covers to define, the first one for A. Let

$$\mathcal{A} := \{B_{\epsilon(x_o)}(x_o) : x_o \in A\}.$$

This open cover has a finite subcover

$$\{B_{\epsilon(x_j)}(x_j) : j = 1,\,\ldots,\,M\}, \ \text{ where each } \ x_j \in A.$$

For each $j = 1,\,\ldots,\,M$, we define the following "big" open cover of B.

$$\{B_{\epsilon(x_j)}(y) : y \in B\}.$$

Each of these covers has a finite subcover

$$\mathcal{B}_j := \{B_{\epsilon(x_j)}(y_{j,k}) : k = 1,\,\ldots,\,N_j\}.$$

Finally we construct the following *finite* collection, which will lead to the desired finite subcover:

$$\{B_{\epsilon(x_j)}(x_j) \times B_{\epsilon(x_j)}(y_{j,k}) : j = 1,\,\ldots,\,M,\ k = 1,\,\ldots,\,N_j\}.$$

We note that each $B_{\epsilon(x_j)}(y_{j,k})$ is contained in some $B_{r_i(x_j)/\sqrt{2}}\big(y_i(x_j)\big)$ in $\mathcal{F}_{x_j}$. Here, the subscript i depends on j and k. Then, since $\epsilon(x_j) \le r_i(x_j)/\sqrt{2}$, it is also true that

$$B_{\epsilon(x_j)}(x_j) \times B_{\epsilon(x_j)}(y_{j,k}) \subseteq B_{r_i(x_j)/\sqrt{2}}(x_j) \times B_{r_i(x_j)/\sqrt{2}}\big(y_i(x_j)\big) \subseteq B_{r_i(x_j)}\big((x_j,\, y_i(x_j)\big) \subseteq U_{j,\,k},$$

for some $U_{j,\,k} \in \mathcal{U}$.

All we need to do now is to check that

$$\{B_{\epsilon(x_j)}(x_j) \times B_{\epsilon(x_j)}(y_{j,k}) : j = 1, \ldots, M,\ k = 1, \ldots, N_j\}$$

is an open cover of $A \times B$, for then the collection $\{U_{j,\,k} : j = 1, \ldots, M,\ k = 1, \ldots, N_j\}$ will be an open cover of $A \times B$ as well.

Let $(x, y) \in A \times B$ be given. Then $x \in B_{\epsilon(x_j)}(x_j)$ for some j. Since $\mathcal{B}_j$ is an open cover of B, $y \in B_{\epsilon(x_o)}(y_{j,k})$ for some $k \in \{1, \ldots, N_j\}$. But then $(x, y) \in B_{\epsilon(x_j)}(x_j) \times B_{\epsilon(x_j)}(y_{j,k})$. This completes the proof.

Remarks

1) We could have proved this Theorem with A and B being compact sets in metric spaces X and Y respectively, instead of Euclidean spaces. But then we would have had to know about building a metric topology on $X \times Y$ out of metrics on X and on Y. This can be done by defining a metric on $X \times Y$ by

$$d_{X\times Y}\big((x, y), (x', y')\big) := \sqrt{d_X(x, x')^2 + d_Y(y, y')^2}.$$

The proof would be done the same way.

2) We could have proved this Theorem with A and B being compact sets in topological spaces X and Y respectively, instead of Euclidean spaces. But then we would have had to know about building a topology on $X \times Y$ out of topologies on X and on Y. Too briefly put, this can be done by letting the topology on $X \times Y$ consist of arbitrary unions of finite intersections of sets of the form $U \times V$, where U is open in X and V is open in Y.

The proof would not be done quite the same way, but would use the same ideas.

3) One application of this Theorem is to show that a k-cell is compact, knowing in advance that a 1-cell is compact.

4) This proof would have been a little easier had we been working with open cubes (here, always with sides parallel to the coordinate axes) instead of open balls, because the Cartesian product of two cubes with the same edge-lengths is a cube. In fact, the proof would also be correct, because it is true that a set U in $\mathbb{R}^k$ is open if and only if for every $x \in U$, there exists $\delta > 0$ such that the cube $Q_\delta(x)$ with center x and edges of length δ is contained in U. This in turn is shown by noting that

$$Q_{\delta/\sqrt{k}}(x) \subseteq B_\delta(x) \subseteq Q_{2\delta}(x).$$