Part 1.

Claim: $x_n := \sqrt{n}$ is not a Cauchy sequence in the metric $d(x,y) := \frac{|x-y|}{1+|x-y|}$

Proof:

$$d(x_n, x_m) = \frac{|\sqrt{n}-\sqrt{m}|}{1+|\sqrt{n}-\sqrt{m}|}$$

Suppose it is a Cauchy sequence. Let $\varepsilon = \frac{1}{2}$. Then there exists $N \in \mathbb{N}$ such that FOR ALL $n \geq N$ and FOR ALL $m \geq N$ we have (assuming $n > m$):

$$d(x_n, x_m) = \frac{\sqrt{n}-\sqrt{m}}{1+\sqrt{n}-\sqrt{m}} < \frac{1}{2}$$

Set $m = N$. Then:

$$\lim_{n\to\infty}\left(\frac{\sqrt{n}-\sqrt{N}}{1+\sqrt{n}-\sqrt{N}}\right) = 1$$

I believe this is a contradiction.

Part 2.

Moreover. Consider an arbitrary sequence $\{x_n\} \subset \mathbb{R}$ that is a Cauchy sequence in the metric $d(x,y)$. Then, given $\varepsilon > 0$ we have (for all n, m greater than some N):

$$\frac{|x_n - x_m|}{1+|x_n - x_m|} < \frac{\varepsilon}{1+\varepsilon}$$

and the RHS can be made arbitrarily small. But this inequality implies that:

$$|x_n - x_m| < \varepsilon,$$

i.e. every Cauchy sequence in the metric $d(x,y)$ is also a Cauchy sequence in the standard metric.