The object of this note is to introduce you to complex numbers (notation: $\mathbb{C}$) and to stress the importance of absolute values in complex numbers. Absolute values allow us to use the same arguments with inequalities that we use with real numbers, on complex numbers! Thus limits and continuity are defined with the same symbols; convergence of sequences and series look the same! Some things are not the same though because the Order Axioms do not hold for the Complex Number System. Since the Completeness Axiom for the Real Numbers is expressed in terms of order, that Axiom does not make sense in the context of complex numbers. However, a substitute is true: "Every Cauchy sequence of complex numbers converges to a limit that is a complex number."

Complex numbers defined in terms of real numbers and i, then in two other ways

We define the *complex number* $z := x + iy$ where x and y are real numbers and where the symbol i is called *the imaginary unit* and has the property $i^2 = -1$. All the usual operations are then carried out: commutativity and associativity are taken for granted. Now i is a solution of the equation $z^2 + 1 = 0$, so i cannot be a real number. We need to say what we mean by *equality* of complex "numbers:" $z = x + iy$ and $w = u + iv$ are equal (by definition) if and only if $x = u$ and $y = v$.

This way of defining complex numbers is the most practical one, but it makes most people very uncomfortable; not only is i an *imaginary* object it is an *imaginary and mysterious and unbelievable* unit! So sometimes the complex numbers are introduced as a *conventional* way of writing out points (x, y) in the "plane" $\mathbb{R}^2$. Thus $z = x + iy$ corresponds to the point (x, y), and vice versa. This is definitely better from a geometric perspective. But then the operation of multiplication has to be *defined* for pairs of points – and the definition looks weird:

$$(x, y)(u, v) := (xu - yv,\, xv + yu) \text{ instead of } (x + iy)(u + iv) = xu + ixv + iyu + i^2yv = xu - yv + i(xv + yu).$$

Recall that we call two points in the plane equal if and only if their corresponding coordinates are equal. In terms of points, i is represented by $(0, 1)$.

There is yet another way of defining the complex number system in terms of special 2×2 matrices. The matricial way is understandable for those who know that number systems are defined mathematically by sets of axioms. It is even more understandable for those who know about *isomorphic* systems: systems that can be put into one-to-one correspondence in such a way that the operations in the two systems are preserved by the correspondence. In particular, it can be shown that, with the definitions of addition and multiplication given here, that Axioms (0) – (10) for $\mathbb{R}$ are true for $\mathbb{C}$ as well.

We will use the first definition. For those who are bothered by it, we'll include the matricial and point approaches as we go.

The matrix correspondence is this: $z = x + iy$ corresponds to the matrix $Z := \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$. Conversely, any matrix with this special form corresponds to the point (x, y) and to $z = x + iy$. We can write any of our special matrices as

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} = x \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + y \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} =: xI + yJ.$$

Here, I is the 2×2 identity matrix and J is the 2×2 matrix $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $J^2 = -I$. Suddenly we have an object defined using real numbers only (and matrices) that satisfies the equation $Z^2 + I = 0$. Well, it's really a 2×2 matrix equation. I hope it takes away any "mysterious and unbelievable" part of your mastering of complex numbers.

Very Important: You should verify that if $z = x + iy$ and $w = u + iv$, and if Z and W are the corresponding matrices, then the matrix that corresponds to zw is the same as the product matrix ZW (formed by following the rules for multiplying matrices). Then, once you know that $zw = wz$, it will follow that $ZW = WZ$.

Recall that we call two matrices equal if and only if they have the same dimensions and their corresponding entries are equal. **Important:** when $y = 0$, so that $Z = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = xI$, we can think of Z as though it were a real number.

The conjugate of a complex number, and a complex number's Real and Imaginary parts

The *conjugate* of a complex number $z = x + iy$ is *denoted* $\bar{z}$ and *defined* by the formula

(1) $$\bar{z} := x - iy.$$

We notice right away that $\bar{\bar{z}} = z$. An **important formula** involving conjugates is

(2) $$\overline{zw} = \bar{z}\,\bar{w}.$$

Its proof will be a good review of multiplying complex numbers:

$$\overline{zw} = \overline{(x+iy)(u+iv)} = \overline{xu - yv + i(xv + yu)} = xu - yv - i(xv + yu),$$

and

$$\bar{z}\bar{w} = (x - iy)(u - iv) = (x + i(-y))(u + i(-v)) = xu - (-y)(-v) + i(x(-v) + (-y)u) = xu - yv - i(xv + yu),$$

which argees with $\overline{zw}$.

Since $\bar{z} = x + i(-y)$, the matrix corresponding to $\bar{z}$ is $\bar{Z} := \begin{pmatrix} x & y \\ -y & x \end{pmatrix} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}^T$, the transpose of the matrix corresponding to z. The *point* corresponding to $\bar{z}$ is the reflection of the point corresponding to z in the x-axis.

The *Real part* of $z = x + iy$, denoted $\mathbf{Re}\ z$, is *defined*, in terms of the conjugate of z, by

(3) $$\mathbf{Re}\ z := \frac{z + \bar{z}}{2};\ \text{ it's important to note that } \mathbf{Re}\ (x + iy) = x.$$

In matrix terms, we can define $\mathbf{Re}\ Z$ similarly: $\mathbf{Re}\ Z = \frac{Z + Z^T}{2}$. In this context, it's important to note that

$$\mathbf{Re}\begin{pmatrix} x & -y \\ y & x \end{pmatrix} = \frac{1}{2}\begin{pmatrix} x & -y \\ y & x \end{pmatrix} + \frac{1}{2}\begin{pmatrix} x & y \\ -y & x \end{pmatrix} = xI.$$

In terms of points in the plane, $\mathbf{Re}\ (x, y) = (x, 0)$, the projection of the point (x, y) on the x-axis.

The Imaginary part is defined similarly, but its definition is complicated by the presence of the i with the y, so we just divide it out, or multiply by $1/i$.

ALARM!! What is $1/i$? The reciprocal, z^{-1}, of a "number" $z \neq 0$ is the solution w of the equation $wz = 1$. *A* solution of $wi = 1$ is $w = -i$. Is this solution unique?

The *Imaginary part* of $z = x + iy$, denoted $\mathbf{Im}\ z$, is *defined*, in terms of the conjugate of z, by

(4) $$\mathbf{Im}\ z := \frac{z - \bar{z}}{2i};\ \text{ it's important to note that } \mathbf{Im}\ (x + iy) = y.$$

In matrix terms, we can define $\mathbf{Im}\ Z$ somewhat similarly: $\mathbf{Im}\ Z = \frac{Z - Z^T}{2}(-J)$. We did not "divide!" We multiplied by the inverse of J, which is $-J$. In this context, it's important to note that

$$\mathbf{Im}\begin{pmatrix} x & -y \\ y & x \end{pmatrix} = \left(\frac{1}{2}\begin{pmatrix} x & -y \\ y & x \end{pmatrix} - \frac{1}{2}\begin{pmatrix} x & y \\ -y & x \end{pmatrix}\right)(-J) = yJ(-J) = yI.$$

In terms of points in the plane, $\mathbf{Im}\ (x, y) = (0, y)(0, -1) = (y, 0)$, the projection of the point (x, y) on the y-axis, then rotated to the x-axis.

The absolute value of a complex number, and its "behavior"

We *define* the absolute value of a complex number z, denoted $|z|$, using the conjugate, by the formula

(5) $$|z|^2 := z\bar{z}, \quad \text{so that} \quad |z| = \sqrt{z\bar{z}}.$$

When we have $z = x + iy$, this means that

$$|x+iy|^2 = (x+iy)(x-iy) = x^2 - (iy)^2 = x^2 + y^2, \quad \text{so we have} \quad |x+iy| = \sqrt{x^2+y^2}.$$

That is, **the absolute value of a complex number is the distance from the number, thought of as a "point," to the origin.** In the matrix context,

$$Z\bar{Z} = ZZ^T = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}\begin{pmatrix} x & y \\ -y & x \end{pmatrix} = \begin{pmatrix} x^2+y^2 & 0 \\ 0 & x^2+y^2 \end{pmatrix} = (x^2+y^2)I = |z|^2 I$$

In the "point" context, $(x,\, y)(x,\, -y) = (x^2 - y(-y),\, yx + x(-y)) = (x^2+y^2,\, 0)$.

When a and b are real numbers, $|ab| = |a||b|$. We can show this is true in the complex-number context too. We try to work with the *square* of the absolute value. We just have to remember to take the square root when done! By definition,

(6) $$|zw|^2 = zw \cdot \overline{zw} = z\bar{z} \cdot w\bar{w} = |z|^2|w|^2, \quad \text{so} \quad |zw| = |z||w|.$$

When a and b are real numbers, $|a+b| \le |a| + |b|$. To show this, the **triangle inequality,** is true in the complex case, we need to notice that, for $z = x + iy$,

(6a) $$(\pm\mathbf{Re}\ z \le)\ |\mathbf{Re}\ z| = |x| \le \sqrt{x^2+y^2} = |z|.$$

We may as well also notice that

(6b) $$(\pm\mathbf{Im}\ z \le)\ |\mathbf{Im}\ z| = |y| \le \sqrt{x^2+y^2} = |z|.$$

Now we can begin proving the triangle inequality:

$$|z+w|^2 = (z+w)(\,\overline{z+w}\,) = (z+w)(\,\bar{z}+\bar{w}\,) = z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} = |z|^2 + (z\bar{w} + w\bar{z}) + |w|^2.$$

The term $z\bar{w} + w\bar{z}$ can be rewritten as $z\bar{w} + \overline{z\bar{w}}$, so it has the form of a complex number plus its conjugate. Thus $z\bar{w} + w\bar{z} = 2\mathbf{Re}\ z\bar{w}$, so that, using (6$a$) to obtain the inequality $\mathbf{Re}\ z\bar{w} \le |z\bar{w}|$,

$$|z+w|^2 = |z|^2 + 2\mathbf{Re}\ z\bar{w} + |w|^2 \le |z|^2 + 2|z\bar{w}| + |w|^2 = |z|^2 + 2|z||w| + |w|^2 = (|z| + |w|)^2.$$

Thus the triangle inequality holds in the complex context:

(7) For all complex numbers z and w, $|z+w| \le |z| + |w|$: **The triangle inequality**

Indeed, it is really in this case that "triangle" inequality becomes meaningful!

In the proof of the triangle inequality, we used this (did you notice?): $|\bar{w}| = |w|$. Can you prove it?

Limits of complex–valued functions of a complex variable and complex sequences and series

The point of what we have developed so far is that we can now use a lot of definitions and theorems about real-valued functions, sequences and series in the context of complex numbers.

(8) **Definition:** *A sequence of complex numbers, or a complex sequence, is a function* $z : \mathbb{N} \to \mathbb{C}$. Usually we write z_n instead of $z(n)$.

(9) **Definition:** *A complex sequence* $\{z_n\}$ *is a (complex) null sequence if for all* $\epsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that for all natural numbers* $n \geq N$, $|z_n| < \epsilon$. We notice that this definition is exactly the same as the definition for real sequences, except for the nature of the terms in the sequence.

(10) **Definition:** *For all* $\epsilon > 0$ *and all* $z_o \in \mathbb{C}$, *the set*

(11) $$B_\epsilon(z_o) := \{z \in \mathbb{C} : |z - z_o| < \epsilon\} \text{ is called the } \epsilon\text{–neighborhood of } z_o.$$

Geometrically, this is the open disc with radius ϵ and center z_o.

Now we turn the idea around: we next define "limit" for complex–valued functions of a complex variable. Then since we can think of the real numbers as a subset of the complex numbers, we can apply the same definition to real–valued functions of a real variable!

(12) **Definition:** Suppose that $\Omega \subseteq \mathbb{C}$ is a non-empty set, $f : \Omega \to \mathbb{C}$ is a complex-valued function defined on Ω, L is a complex number and $z_o \in \mathbb{C}$. We say that

$$\lim_{\substack{z \to z_o \\ z \in \Omega}} f(z) = L \text{ if}$$

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall z \in \mathbb{C})([z \in \Omega \ \& \ 0 < |z - z_o| < \delta] \Rightarrow |f(z) - L| < \epsilon).$$

This looks exactly like the definition of limit for real-valued functions of a real variable. **It is very important to notice that in this definition the variable z is never allowed to equal z_o!** By the way, if the definition is true for some Ω, f, L and z_o, what does the definition "say" if z_o is not in Ω?

The next definition addresses continuity at a point. It can again be used in the "real" context.

(13) **Definition:** Suppose that $\Omega \subseteq \mathbb{C}$ is a non-empty set, $f : \Omega \to \mathbb{C}$ is a complex-valued function defined on Ω, L is a complex number and $z_o \in \Omega$. We say that $f(z)$ is continuous in Ω at z_o if

$$\lim_{\substack{z \to z_o \\ z \in \Omega}} f(z) = f(z_o).$$

We will need to use the idea of the *derivative in the complex sense.*

(14) **Definition:** Suppose that $\Omega \subseteq \mathbb{C}$ is a non-empty set, $f : \Omega \to \mathbb{C}$ is a complex-valued function defined on Ω, L is a complex number and $z_o \in \Omega$. We further suppose that for some $d > 0$, all complex numbers z such that $|z - z_o| < d$ also belong to Ω. We say that $f(z)$ is *complex differentiable* in Ω at z_o if there is a complex number $f'(z_o)$ such that

$$\lim_{\substack{z \to z_o \\ z \in \Omega}} \frac{f(z) - f(z_o)}{z - z_o} = f'(z_o).$$

The expression $F(z) := \frac{f(z)-f(z_o)}{z-z_o}$, **which is defined only for those $z \in \Omega$ that are not equal to z_o**, is called a *difference quotient.* The definition requires that

$$\lim_{\substack{z \to z_o \\ z \in \Omega}} F(z) = f'(z_o).$$

This means that the limit of the difference quotient exists, and that the limit has the value $f'(z_o)$. We will frequently replace z here by $z_o + h$, where h is a complex number close to zero, e.g., $|h| < d$ in the definition. In this case, we'll be looking for

$$\lim_{\substack{h \to 0 \\ z_o + h \in \Omega}} \frac{f(z_o + h) - f(z_o)}{h}.$$

Next, suppose that $\{z_n\}$ is a sequence of complex numbers. We will now define limit for complex sequences.

(15) **Definition:** Suppose that $\{z_n\}$ is a sequence of complex numbers and that $L \in \mathbb{C}$. We say that $\{z_n\}$ converges to L, denoted

$$\lim_{n\to\infty} z_n = L, \quad \text{if } \{z_n - L\} \text{ is a null sequence.}$$

This is, "in logic," $(\forall\epsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n \geq N \Rightarrow |z_n - L| < \epsilon)$.

The definition looks the same, and behaves the same, as its real counterpart, even though "absolute value" means something else here.

(16) **A handy Theorem:** $z_n \to L$ if and only if $\mathbf{Re}\ z_n \to \mathbf{Re}\ L$ and $\mathbf{Im}\ z_n \to \mathbf{Im}\ L$.

Proof: $|z_n - L|^2 = |\mathbf{Re}\ z_n - \mathbf{Re}\ L|^2 + |\mathbf{Im}\ z_n - \mathbf{Im}\ L|^2$.

This theorem has this useful point: convergence of complex sequences is really the same thing as the convergence of two real sequences, the sequences of real and imaginary parts.

The following is for later reading. . .

There are some other useful points. One is that the definition of convergence of a series is exactly the same as in the real case: a series $\sum_{n=1}^{\infty} z_n$ of complex numbers converges if and only if the sequence $\{s_n\}$ of partial sums converges, where $s_n := \sum_{k=1}^{n} z_k$. You should state this as a Definition. You should also define *absolutely convergent* for series of complex numbers.

You can now prove the absolute convergence theorem. Please do so; here is the Theorem:

Theorem: *If the series $\sum_{n=1}^{\infty} z_n$ of complex numbers converges absolutely, then $\sum_{n=1}^{\infty} z_n$ converges.*

We will use this theorem a lot in our study of power series, for we can use our real-valued theorems on the series of absolute values!

Example: *For every natural number k the series $\sum_{n=0}^{\infty} n^k z^n$ converges absolutely if $|z| < 1$ and diverges if* $|z| \geq 1$. To prove this we look at $|n^k z^n| = n^k |z|^n$ and we apply the Ratio Test:

$$\frac{(n+1)^k |z|^{n+1}}{n^k |z|^n} = \frac{(n+1)^k |z|}{n^k} = \left(\frac{n+1}{n}\right)^k |z| = \left(1 + \frac{1}{n}\right)^k |z| \to |z|.$$

Convergence follows if $|z| < 1$. The series diverges if $|z| \geq 1$ not by the Ratio Test but because the terms do not tend to zero.