Here is a *third* proof of the "continuous implies boundedness" Theorem. This is approximately what we did in class.

Theorem: *If f is continuous on a closed and bounded interval then f is bounded.*

Proof: Let the (nonempty!) interval be denoted $[a, b]$. We will use contradiction. Thus for every positive natural number n there exists $x_n \in [a, b]$ such that $|f(x_n)| > n$. We will show later that there are infinitely many *distinct* x_n's, which makes the set

$$S := \{y \in \mathbb{R} : (\exists n \in \mathbb{N})(y = x_n)\} \text{ an infinite and bounded set.}$$

By the Bolzano–Weierstrass Theorem, there exists a point p that is a point of accumulation for S. We will show later that $p \in [a, b]$. We will also show later that every neighborhood of a point of accumulation for S contains infinitely many points of S that are different than p. We will treat these three things, to be proved later, as true.

Since $p \in [a, b]$, f is continuous at p. Taking $\epsilon = 1$ in the definition of continuity of f at p we obtain $\delta > 0$ such that whenever $|x - p| < \delta$ and $x \in [a, b]$, $|f(x) - f(p)| < \epsilon = 1$. Therefore $|f(x)| < |f(p)| + 1$ whenever $|x - p| < \delta$ and $x \in [a, b]$. Since there are infinitely many distinct x_n's in $(p - \delta, p + \delta)$ there are infinitely many n such that $|f(x_n)| < |f(p)| + 1$. But since $|f(x_n)| > n$, there must be infinitely many n such that $n < |f(p)| + 1$. This is a contradiction. Thus f is indeed bounded on $[a, b]$.

The statements left for later proof

There are infinitely many distinct x_n's.

Suppose not. Then every x_n is in the finite set consisting of certain ones of the x_n's. Suppose there are exactly K of these x_n's, say $x_{n_1}, \ldots, x_{n_K}$, that are distinct. Hence every $n \in \mathbb{N}$ is in one of the sets

$$E_j := \{j \in \mathbb{N} : x_j = x_{n_j}\}, \quad \text{for } 1 \le j \le K.$$

The set $\{|f(x_{n_1})|, \ldots, |f(x_{n_K})|\}$ is finite so it has a largest element, say $|f(x_{n_J})|$, where $1 \le J \le K$.

But then for every $n \in \mathbb{N}$, $n < |f(x_n)| \le |f(x_{n_J})|$ which asserts that $\mathbb{N}$ is bounded above. Since this is false, our assumption that only finitely many distinct x_n's is false, so the desired truth value has been established.

To show: $p \in [a, b]$.

If p is a point of accumulation of a set S and $S \subseteq T$, then p is also a point of accumulation for T, because if a neighborhood of p contains a point of S that is different than p, that same point is a point of T that is different than p. Closed intervals are among the examples of sets that we have shown to be closed sets. Thus $[a, b]$ is a closed set. We have proved that a set is closed if and only if it contains all of it points of accumulation. Since p is a point of accumulation for $S \subseteq [a, b]$, p is a point of accumulation for $[a, b]$. Thus $p \in [a, b]$.

Every neighborhood of a point of accumulation for S contains infinitely many points of S.

Suppose not. Let p denote a "contrary" point of accumulation. Then there exists a neighborhood $(p - h, p + h)$ of p such that only finitely many points, $s_1, \ldots, s_L$, of S that differ from p are in $(p - h, p + h)$. There must be at least one such point, by the definition of point of accumulation. Then the set $\{|p - s_1|, \ldots, |p - s_L|\}$ is a finite set of positive numbers that therefore has a least (positive!) element, call it δ. But then $(p - \delta, p + \delta)$ contains *no* points of S. This contradicts the definition of point of accumulation.

A remark Suppose your mind asked "*Why* are there only finitely many natural numbers n less than $|f(p)| + 1$?"

Suppose that the set $C := \{n \in \mathbb{N} : n < |f(p)| + 1\}$ *is* infinite. We know that there exists a natural number $L > |f(p)| + 1$. Since C is infinite, it is nonempty, and when we remove an element from it the elements left behind still comprise an infinite set. Indeed, if we do this L times, there will *still* be infinitely many elements left in the partly depleted set C. We will remove from C its smallest element, n_1, then the smallest element, n_2, of $C \setminus \{n_1\}$, and so on until we have removed the L elements $n_1 < n_2 < \cdots < n_L$ from C. Now $n_{L+1} < |f(p)| + 1$ because $n_{L+1} \in C$. Now we notice that

$$L > n_{L+1} = (n_{L+1} - n_L) + (n_L - n_{L-1}) + \cdots + (n_2 - n_1) + n_1 \ge L + 1, \quad \text{a contradiction.}$$