Note 10/29: This list is from another course. It will be edited to better reflect Math 5583. *It is not a complete list, nor is the absence of a Definition or Theorem indicative that the Definition or Theorem will not appear on the next Test!!* The list is here on 10/29 because of *uniform convergence*. See pages 7 and 8.

Definition: A set $S \subseteq \mathbb{R}$ *has a limit point* if there exists $x_o \in \mathbb{R}$ such that for all $\delta > 0$, there exists $x \in \mathbb{R}$ such that $x \in S \ \wedge \ 0 < |x - x_o| < \delta$. That is, there is a real number that has members of the set S arbitrarily close to it that are not equal to it.

Definition: (The sets that model finite sets) For each $n \in \mathbb{N}$, let $S_n := \mathbb{N} \cap \{ x \in \mathbb{R} : x < n \}$. E.g., S_0 is empty, $S_1 = \{0\}$, $S_2 = \{0, 1\}$, and so on.

Definition: A set S (that is a subset of some universal set) is *finite,* or S *is a finite set* if S can be put into one-to-one correspondence with one of the sets S_n. That is, S *is finite* if there exists $n \in \mathbb{N}$ and if there exists a function $\gamma : S \to S_n$ such that γ is both one-to-one and onto.

Definition: Given $n \in \mathbb{N}$, a set S *has* n *elements* if S can be put into one-to-one correspondence with S_n.

Definition: Given $n \in \mathbb{N}$, a set S *has more than* n *elements* if there exists a proper subset T of S that has n elements. (A set T is a *proper subset of* S if $T \subseteq S$ and $T \neq S$.)

Definition: A set S (that is a subset of some universal set) is *infinite* if S is not finite.

Ways we can tell a set is infinite

Theorem: *A set* $S \subseteq \mathbb{R}$ *is infinite if* S *is unbounded.*

Theorem: *A set* S *is infinite if for all* $n \in \mathbb{N}$ S *has more than* n *elements.*

Theorem: *A set* S *is infinite if there exists a function* $\gamma : S \to \mathbb{N}$ *such that* γ *is onto.*

Theorem: *A set* S *is infinite if there exists a function* $\gamma : \mathbb{N} \to S$ *such that* γ *is one-to-one.*

Definition: A set S is *denumerable* if there exists a function $\gamma : \mathbb{N} \to S$ such that γ is both onto and one-to-one. (That is, S can be put into one-to-one correspondence with $\mathbb{N}$. If it is more convenient, we can work with $\gamma : S \to \mathbb{N}$ such that γ is both one-to-one and onto.)

Definition: A set S is *countable* if S is finite or denumerable.

Definition: A set S is *uncountable* if S is not countable.

Examples: $\mathbb{N}$ is denumerable, the even members of $\mathbb{N}$ form a set that is denumerable. $\mathbb{Z}$ and $\mathbb{Q}$ are denumerable. $\mathbb{R}$ is uncountable, i.e., not countable. Finite sets are countable.

Theorem: *Every infinite set contains a proper subset that is countable.*
This Theorem says that "countable" is the smallest-sized infinity.

Theorem: *Every bounded infinite subset of* $\mathbb{R}$ *has a limit point.*

The two following Theorems are (together) called "Sequences are Good Enough" (the real case)

Theorem: *If* $f : D \to \mathbb{R}$ *and if* f *is continuous at* $x_o \in D$ *then, for all sequences* $\{x_n\}$ *that consist entirely of points of* D, $x_n \to x_o \Longrightarrow f(x_n) \to f(x_o)$.

Theorem: *If* $f : D \to \mathbb{R}$ *and* $x_o \in D$ *and if for all sequences* $\{x_n\}$ *that consist entirely of points of* D, $x_n \to x_o \Longrightarrow f(x_n) \to f(x_o)$, *then* f *is continuous at* x_o.

Theorem: (Boundedness Theorem) *If* $f : [a, b] \to \mathbb{R}$ *is continuous then* f *is bounded on* $[a, b]$. That is, there exists $A \in \mathbb{R}$ such that $|f(x)| \leq A$ for all $x \in [a, b]$.

Theorem: (Extreme Value Theorem) *If* $f : [a, b] \to \mathbb{R}$ *is continuous then there exist* α *and* β *in* $[a, b]$ *such that* $f(\alpha) \leq f(x) \leq f(\beta)$ *for all* $x \in [a, b]$. That is, f takes on maximum and minimum values in $[a, b]$.

Theorem: *If* I *is an interval and* $f : I \to \mathbb{R}$ *is continuous and one-to-one then* f *is strictly monotone on* I.

Definition: A set $S \subseteq \mathbb{R}$ is *open* if for every $x_o \in S$ there exist real numbers a and b (which need not belong to the set S) such that $a < x_o < b$ and such that $(a, b) \subseteq S$. That is, *all* points x that are close enough to x_o are in S.

Definition: A set $S \subseteq \mathbb{R}$ is *closed* if its complement is open.

Examples: Open intervals are open, closed intervals are closed. The interval $[0, 1)$ is neither open nor closed. The set consisting of the reciprocals of the positive integers is neither open nor closed. When we add 0 to the last set, the new set is closed.

Definition: A set N is a *neighborhood* of a point x_o if there exists $\delta > 0$ such that $(x_o - \delta, x_o + \delta) \subseteq N$. (Since $(x_o - \delta, x_o + \delta)$ is an open set and $x_o \in (x_o - \delta, x_o + \delta)$ and $(x_o - \delta, x_o + \delta) \subseteq N$, we can say a set N is a neighborhood of a point x_o if and only if the set N contains an open set that contains x_o.)

Definition: A point x_o is a *boundary point* of a set S if every neighborhood of x_o contains a point in S and a point not in S. (The point x_o can be the point in S or the point not in S.) (Remark: A boundary point of S is also a boundary point of S^c, but a limit point of S may or may not be a limit point of S^c. For example, if $S = (0, 1)$, then 0 and $1/2$ are limit points of S, 0 is a limit point of S^c, and $1/2$ is not a limit point of S^c.)

Definition: A point x_o is an *interior point* of a set S if there exists $\delta > 0$ such that $(x_o - \delta, x_o + \delta) \subseteq S$.

Definition: A point x_o is an *isolated point* of a set S if $x_o \in S$ and if there exists $\delta > 0$ such that $S \cap (x_o - \delta, x_o + \delta) = \{x_o\}$.

Definition: Suppose that $f(x)$ is defined on an open interval (a, b), and that $x_o \in (a, b)$. Then f *is differentiable at* x_o if the limit of the *difference quotient,* $\dfrac{f(x) - f(x_o)}{x - x_o}$, exists as x tends to x_o. In other words, f *is differentiable at* x_o, or, *the derivative of* f *at* x_o *exists,* if $\lim\limits_{x \to x_o} \dfrac{f(x) - f(x_o)}{x - x_o}$ exists.

Notation: If f is differentiable at x_o, we write $f'(x_o)$ or $\left.\dfrac{df}{dx}\right|_{x=x_o}$ for the limit of the difference quotient, $\dfrac{f(x) - f(x_o)}{x - x_o}$, as x tends to x_o. That is,

$$f'(x_o) = \lim_{x \to x_o} \frac{f(x) - f(x_o)}{x - x_o} = \left.\frac{df}{dx}\right|_{x=x_o}$$

Examples: $\frac{x^2 - x_o^2}{x - x_o} = x + x_o \to 2x_o$; $\frac{x^3 - x_o^4}{x - x_o} = x^2 + x x_o + x_o^2 \to 3x_o^2$; $\frac{x^n - x_o^n}{x - x_o} = x^{n-1} + x^{n-2}x_o + \cdots + x x_o^{n-2} + x_o^{n-1} \to n x_o^{n-1}$.

Theorem: *If* f *is differentiable at* x_o, *then* f *is continuous at* x_o.

Theorem: (The Product Rule) *If* f *and* g *are differentiable at* x_o, *then* $h(x) := f(x)g(x)$ *is differentiable at* x_o *and*

$$h'(x_o) = f'(x_o)g(x_o) + f(x_o)g'(x_o).$$

Theorem: *If* f *and* g *are differentiable at* x_o, *then* $h(x) := af(x) + bg(x)$ *is differentiable at* x_o *and*

$$h'(x_o) = af'(x_o) + bg'(x_o).$$

Theorem: (The Chain Rule) *If* f *is differentiable at* $g(x_o)$, *and* g *is differentiable at* x_o, *then* $h(x) := f(g(x))$ *is differentiable at* x_o *and*

$$h'(x_o) = f'(g(x_o))g'(x_o).$$

Theorem: *For all integers* n, $\left.\dfrac{d}{dx} x^n\right|_{x=x_o} = n x_o^{n-1}$, *provided that, if* $n < 0$, $x_o \neq 0$; *if* $n = 0$, *the formula is to be interpreted as having the value zero for all* x_o.

Theorem: *If f is continuous on $[a, b]$ and differentiable on (a, b) and the maximum or minimum value of $f(x)$ occurs at $x_o \in (a, b)$, then $f'(x_o) = 0$.*

Theorem: (Rolle's Theorem, version 0) *If f is continuous on $[a, b]$ and differentiable on (a, b) and $f(a) = 0 = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.*

Theorem: (Rolle's Theorem, full version) *If f is continuous on $[a, b]$ and differentiable on (a, b) and $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.*

Theorem: (The Mean Value Theorem (MVT)) *If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.*

Theorem: (The Cauchy Mean Value Theorem) *If f and g are continuous on $[a, b]$ and differentiable on (a, b) and $g'(x) \neq 0$ anywhere in (a, b), then there exists $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{g(b) - g(a)}$.*

(An application of the MVT) Theorem: *A real-valued function $f(x)$ is constant on an interval (a, b) if and only if $f'(x) = 0$ for all $x \in (a, b)$.*

• **Theorem: (A Theorem of Darboux)** *If f is differentiable on (a, b) and $a < c < d < b$, and v is a number strictly between $f'(c)$ and $f'(d)$, then there exists $\gamma \in (c, d)$ such that $f'(\gamma) = v$.*
In other words, derivatives have the Intermediate Value Property, just as continuous functions do.

Remarks on finding maxima and minima

Definition: A function defined on an interval containing x_o *has a local extremum at* x_o it there exists a neighborhood of x_o in which $f(x) \leq f(x_o)$ (a local maximum) or in which $f(x) \geq f(x_o)$ (a local minimum). Here, the x's are those that are in the neighborhood and in the domain of f. In other words, there exists $\delta > 0$ such that for all $x \in D \cap (x_o - \delta, x_o + \delta)$, $f(x) \leq f(x_o)$ (a local maximum) or there exists $\delta > 0$ such that for all $x \in D \cap (x_o - \delta, x_o + \delta)$, $f(x) \geq f(x_o)$ (a local minimum).

Theorem: *If f is continuous on $[a, b]$ and differentiable on (a, b), and if $f'(a) > 0$ and $f'(b) < 0$ then the maximum value of $f(x)$ on $[a, b]$ is assumed at some point $c \in (a, b)$.*

On Series of real numbers

"**Definition:** " A *series* is a mathematical object that involves a sequence of *terms* a_n, the symbol $\sum_{n=0}^{\infty} a_n$, the associated *sequence of partial sums,* $\{s_n\}_{n=0}^{\infty}$, where $s_n := \sum_{k=0}^{n} a_k$, and the *behavior* of the sequence of partial sums, namely, convergent or divergent, and where divergence can be bounded oscillating, unbounded oscillating, or divergence to $\pm\infty$. There are also types of convergence, namely absolute and conditional convergence.

Remark: A series does not have to begin with $n = 0$. It can begin with any integer, positive, negative or zero. We'll state definitions and theorems using "$n = 0$," but they apply equally well to series starting anywhere. It is quite common to see series starting with $n = 1$. The partial sums of a given series always begin with the same n the terms of the series begin with.

Definition: A series $\sum_{n=0}^{\infty} a_n$ *converges,* or *is convergent,* if the sequence of partial sums converges to a finite limit, s. The limit s, if it exists, is called the *sum* of the series, and we then can write $\sum_{n=0}^{\infty} a_n = s$. If the sequence of partial sums does not converge to a finite limit, we say the series *diverges,* or *is divergent.* Please note that, just because we write down the symbol $\sum_{n=0}^{\infty} a_n$, we do not mean that the series converges (i.e., that the sequence of partial sums has a finite limit)! However, when we write an equation such as $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$, we do assert convergence. Of course, we still have to be able to **prove** that convergence happens, and verify the limit.

Remark: The convergence of a series has nothing to do with the first N terms; the series can start anywhere after the first one without affecting convergence or divergence!

Theorem: *If the series $\sum_{n=0}^{\infty} a_n$ converges, then $a_n \to 0$.*

(cautionary)**Example:** The terms of the series $\sum_{n=1}^{\infty} 1/n$ converge to zero, but the series diverges to $+\infty$, so the converse of the preceding theorem is false.

Example: The series $\sum_{n=1}^{\infty} 1/(n(n+1))$ converges to 1; its partial sums are given by the formula $s_n = 1 - 1/(n+1)$. This is a *telescoping* series because when we rwrite the terms $1/(n(n+1))$ as partial fractions, we get $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, which cancel partly with adjacent terms, leaving only the first part of the first term and the last part of the last term.

Example: The series $\sum_{n=1}^{\infty} x^n$ is probably the most important one – it is called the Geometric Series. Its partial sums are given by two formulas, depending on where x is, relative to 1. Here they are:

$$\begin{aligned} s_n = s_n(x) = \sum_{k=0}^{n} x^k &= n+1, \quad \text{if } x = 1; \\ &= \frac{1-x^{n+1}}{1-x}, \quad \text{if } x \neq 1. \\ (&= \frac{x^{n+1}-1}{x-1}, \quad \text{if } x \neq 1, \quad \text{also}) \end{aligned}$$

The formulas for the partial sums of the Geometric Series are used to prove the following Theorem.

Theorem: *The Geometric Series converges if and only if* $|x| < 1$; *it diverges to* $+\infty$ *linearly if* $x = 1$; *it diverges by bounded oscillation if* $x = -1$; *it diverges exponentially if* $x > 1$; *it diverges by unbounded (above and be;ow) oscillation if* $x < -1$. If the Geometric Series converges (so that $|x| < 1$), we have

$$\sum_{n=1}^{\infty} x^n = \frac{1}{1-x}, \quad \text{if and only if} \quad |x| < 1.$$

Since we don't have many exact formulas for partial sums, theorems have been developed that allow us to assert the existence of a sum for a series without knowing what the sum is, by means of the theorems about convergence of monotone sequences, and by means of a wide variety of Comparison-type Theorems. Most Comparison-type theorems are about series with non-negativew or positive terms.

Theorem: The Comparison Test *If* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *are series with non-negative terms and if there exists a positive constant* C *such that for all sufficiently large* n, $a_n \le Cb_n$, *then* $\sum_{n=0}^{\infty} b_n$ *convergent* $\Longrightarrow$ $\sum_{n=0}^{\infty} a_n$ *convergent, and (contrapositive!)* $\sum_{n=0}^{\infty} a_n$ *divergent* $\Longrightarrow$ $\sum_{n=0}^{\infty} b_n$ *divergent.*

Theorem: The Limit Comparison Test *If* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *are series with positive terms and if* $\lim_{n\to\infty} a_n/b_n = L > 0$, *then* $\sum_{n=0}^{\infty} b_n$ *is convergent if and only if* $\sum_{n=0}^{\infty} a_n$ *is convergent.*

Theorem: The Ratio Comparison Test *If* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *are series with positive terms and if* $a_{n+1}/a_n \le b_{n+1}/b_n$ *for all sufficiently large* n, *then* $\sum_{n=0}^{\infty} b_n$ *convergent* $\Longrightarrow$ $\sum_{n=0}^{\infty} a_n$ *convergent, and (contrapositive!)* $\sum_{n=0}^{\infty} a_n$ *divergent* $\Longrightarrow$ $\sum_{n=0}^{\infty} b_n$ *divergent.*

Theorem: The Ratio Test *If* $\sum_{n=0}^{\infty} a_n$ *is a series with positive terms and if* $a_{n+1}/a_n \to r$ *as* $n \to \infty$, *then* $\sum_{n=0}^{\infty} a_n$ *is convergent if* $r < 1$, *divergent if* $r > 1$. *But if* $r = 1$, *the series can diverge or converge; the test gives no information.*

Theorem: The Root Test *If* $\sum_{n=0}^{\infty} a_n$ *is a series with positive terms and if* $a_n^{1/n} \to r$ *as* $n \to \infty$, *then* $\sum_{n=0}^{\infty} a_n$ *is convergent if* $r < 1$, *divergent if* $r > 1$. *But if* $r = 1$, *the series can diverge or converge; the test gives no information.*

Theorem: Raabe's Test *If* $\sum_{n=0}^{\infty} a_n$ *is a series with positive terms and if* $n(1 - \frac{a_{n+1}}{a_n}) \to p$ *as* $n \to \infty$, *then* $\sum_{n=0}^{\infty} a_n$ *is convergent if* $p > 1$, *divergent if* $p < 1$. *But if* $p = 1$, *the series can diverge or converge; the test gives no information.*

Some convergence tests require quantities that appear in the series to be decreasing.

Theorem: Cauchy's Condensation Test *If* $a_n \downarrow 0$ *as* $n \to \infty$, *then* $\sum_{n=0}^{\infty} a_n$ *is convergent if and only if* $\sum_{n=0}^{\infty} 2^n a_{2^n}$ *is convergent.*

Application: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$. (The second series, the "condensed" one) becomes a geometric series in $x = 1/2^{p-1}$ after a bit of algebraic rearranging.

Theorem: The Alternating Series Test *If $a_n \downarrow 0$ as $n \to \infty$, then $\sum_{n=0}^{\infty}(-1)^n a_n$ converges to a sum s, and* $|s - s_n| \leq a_{n+1}$.

Summation by parts formula If $\{a_n\}_{n=n_o}^{\infty}$ and $\{b_n\}_{n=n_o}^{\infty}$ are sequences, let us define $A_n := \sum_{k=n_o}^{n} a_k$ for $n \geq n_o$, and define $A_n := 0$ for $n < n_o$. Then, for each integer $n_1 \geq n_o$,

$$\sum_{n=n_o}^{n_1} a_n b_n = \sum_{n=n_o}^{n_1} A_n(b_n - b_{n+1}) + A_{n_1} b_{n_1+1}.$$

The proofs of the next two theorems can be obtained from the Summation by parts formula and definitions.

- **Theorem: Dirichlet's Test I** *If $\{a_n\}_{n=n_o}^{\infty}$ and $\{b_n\}_{n=n_o}^{\infty}$ are sequences, let us define $A_n := \sum_{k=n_o}^{n} a_k$ for $n \geq n_o$, and define $A_n := 0$ for $n < n_o$. If the sequence $\{A_n\}_{n=n_o}^{\infty}$ is bounded, and the sequence $\{b_n\}_{n=n_o}^{\infty}$ tends to zero as $n \to \infty$, then $\sum_{n=n_o}^{\infty} a_n b_n$ converges.*

- **Theorem: Dirichlet's Test II** *If $\{a_n\}_{n=n_o}^{\infty}$ and $\{b_n\}_{n=n_o}^{\infty}$ are sequences, let us define $A_n := \sum_{k=n_o}^{n} a_k$ for $n \geq n_o$, and define $A_n := 0$ for $n < n_o$. If the series $\sum_{n=0}^{\infty} a_n$ is convergent, and the sequence $\{b_n\}_{n=n_o}^{\infty}$ tends to a limit as $n \to \infty$, then $\sum_{n=n_o}^{\infty} a_n b_n$ converges.*

The Riemann Integral

There are some "jargon" terms we need to define in order to build the Riemann integral.

Definition: A *partition* π of $[a, b]$ consists of finitely many points x_i, arranged in strictly increasing order, such that $a = x_0 < \cdots < x_n = b$. *Notation:* $\pi = \{a = x_0 < \cdots < x_n = b\}$. Also, we can write $\pi\big|[a, b]$ to express the phrase "π is a partition of $[a, b]$." We say that the set $\{x_o, x_1, \ldots, x_n\}$ comprises *the points of* π.

Example: It is handy to let $\Pi_N\big|[a, b]$ denote the "partition of $[a, b]$ into N equal parts" given by $\Pi_N\big|[a, b] := \{a = x_0 < \cdots < x_N = b\}$ where $x_i := a + i\dfrac{b-a}{N}$.

We can also write $\Pi_N\big|[a, b] = \{x_i = a + i\dfrac{b-a}{N} : 0 \leq i \leq n\}$.

When $N = 5$ and $[a, b] = [0, 1]$, we have

$$\Pi_N\big|[0, 1] = \{0 = x_0 < x_1 = 1/5 < x_2 = 2/5 < x_3 = 3/5 < x_4 = 4/5 < x_5 = 1\},$$

which we can simply write as $\{0, 1/5, 2/5, 3/5, 4/5, 1\}$ if we want to!

Definition: The *intervals of a partition* π of $[a, b]$ are the n closed intervals $I_i = I_i(\pi) = [x_{i-1}, x_i]$, $1 \leq i \leq n$.

The theory of the Riemann integral requires (as shown by the following definitions) that the functions it applies to are *bounded,* and that the intervals used are *closed and bounded.*

Definition: Let $f : [a, b] \to \mathbb{R}$ be a bounded function defined on the closed and bounded interval $[a, b]$. The *upper sum of $f(x)$ over the partition π of $[a, b]$* is $\mathcal{U}_\pi = \displaystyle\sum_{i=1}^{n} M_i(x_i - x_{i-1})$, where $M_i = \sup_{[x_{i-1}, x_i]} f(x)$, and

The lower sum of $f(x)$ over the partition π of $[a, b]$ is $\mathcal{L}_\pi = \displaystyle\sum_{i=1}^{n} m_i(x_i - x_{i-1})$, where $m_i = \inf_{[x_{i-1}, x_i]} f(x)$.

Remark: Because $f(x)$ is bounded above and below on $[a, b]$, the sets of upper sums and of lower sums are each bounded, above and below. Moreover, for every partition $\pi\big|[a, b]$, $\mathcal{L}_\pi \leq \mathcal{U}_\pi$.

Definition: Let $f:[a,\,b]\to\mathbb{R}$ be a bounded function defined on the closed and bounded interval $[a,\,b]$. The *upper integral of* $f(x)$ *over* $[a,\,b]$ is denoted, in symbols, by $\overline{\int_a^b} f(x)\,dx$, and is defined by

$$\overline{\int_a^b} f(x)\,dx := \inf_{\pi\,|\,[a,\,b]} \mathcal{U}_\pi .$$

Definition: Let $f:[a,\,b]\to\mathbb{R}$ be a bounded function defined on the closed and bounded interval $[a,\,b]$. The *lower integral of* $f(x)$ *over* $[a,\,b]$ is denoted, in symbols, by $\underline{\int_a^b} f(x)\,dx$, and is defined by

$$\underline{\int_a^b} f(x)\,dx := \sup_{\pi\,|\,[a,\,b]} \mathcal{L}_\pi .$$

That these definitions make sense can be seen from the following important facts about upper and lower sums over different partitions (of the same interval). First, we need a definition.

Definition: If π_1 and π_2 are partitions of the closed and bounded interval $[a,\,b]$, we say that π_2 *is a refinement of* π_1 if the set of points of π_1 is contained in the set of points of π_2. We can write, by abuse of notation, $\pi_1\subseteq\pi_2$.

Examples: Π_{10} is a refinement of Π_5, but not of Π_4.

Theorem: *If* π_2 *is a refinement of* π_1 *and* $f(x)$ *is bounded on the closed and bounded interval* $[a,\,b]$, *then*

$$\mathcal{L}_{\pi_1}\le\mathcal{L}_{\pi_2}\le\mathcal{U}_{\pi_2}\le\mathcal{U}_{\pi_1}.$$

In other words, refining a partition (by adding more points to it) increases lower sums and decreases upper sums.

Definition: If π_1 and π_2 are partitions of the closed and bounded interval $[a,\,b]$, *the common refinement of* π_1 and π_2 is the partition $\pi_{1\&2}$ whose points are made up of all the points of π_1 and of π_2 taken together. We can write, by abuse of notation, $\pi_{1\&2}=\pi_1\cup\pi_2$.

We can put together the definition of common refinement and the theorem about refinements and upper and lower sums to prove the next important fact.

Theorem: *If* π_1 *and* π_2 *are partitions of the closed and bounded interval* $[a,\,b]$, *then*

$$\mathcal{L}_{\pi_1}\le\mathcal{U}_{\pi_2}.$$

In other words, every lower sum is a lower bound for *all* the upper sums, and every upper sum is an upper bound for *all* the lower sums.

These were the facts to justify the definitions of upper and lower integrals. We can now define the Riemann integral.

Definition: Let $f:[a,\,b]\to\mathbb{R}$ be a bounded function defined on the closed and bounded interval $[a,\,b]$. Then f is *Riemann integrable over* $[a,\,b]$ if

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx,$$

and then we write, for the *integral of* f *over* $[a,\,b]$,

$$\int_a^b f(x)\,dx := \underline{\int_a^b} f(x)\,dx \left(= \overline{\int_a^b} f(x)\,dx\right).$$

(non-)Example: The *Dirichlet function* is defined by $D(x)=1$ if x is a rational number, and by $D(x)=0$ if x is an irrational number. Since every interval of positive length contains both rational numbers and irrational numbers, over every partition of any closed and bounded interval $[a, b]$, all the numbers $M_i = 1$ and all the numbers $m_i = 0$. Hence, every upper sum has value $b - a$, and every lower sum has value zero. Thus, by definition, the upper and lower integrals have the same values, so the Dirichlet function is not Riemann integrable over any closed and bounded interval of positive length.

Important: The most useful way to prove that a function, or a class of functions distinguished by some property of its members, is Riemann integrable, is to use the following also theoretically important Lemma.

Lemma: *Let* $f : [a, b] \to \mathbb{R}$ *be a bounded function defined on the closed and bounded interval* $[a, b]$. *Then* f *is Riemann integrable over* $[a, b]$ *if and only if for all* $\epsilon > 0$, *there exists a partition* π *of* $[a, b]$ *such that*

$$\mathcal{U}_{\pi_1} - \mathcal{L}_{\pi_1} < \epsilon.$$

This important lemma is a main tool used to prove several Theorems. The first we shall consider is

Theorem: (Monotone functions are integrable) *If* f *is monotone on a closed and bounded interval* $[a, b]$, *then* f *is Riemann integrable over* $[a, b]$.

The next important Theorem concerns the Riemann integrability of continuous functions. To make effective use of the important Lemma, we need to use something called "uniform continuity: one delta works at all x's"

Definition: A real-valued function f defined on a set $D \subseteq \mathbb{R}$ is *uniformly continuous* on D if for every $\epsilon > 0$ there exists $\delta > 0$ such that for all points x and x' in D, if $|x - x'| < \delta$, then $|f(x) - f(x')| < \epsilon$.

Exercise: Prove that, if f is uniformly continuous on D, then f is continuous on D.

The next theorem has practical as well as theoretical importance.

Theorem: *If* f *is continuous on a closed and bounded interval* $[a, b]$, *then* f *is uniformly continuous on* $[a, b]$.

Theorem: (Continuous functions are integrable) *If* f *is continuous on a closed and bounded interval* $[a, b]$, *then* f *is Riemann integrable over* $[a, b]$.

• **Definition:** A set Z is a *set of (Lebesgue) measure zero* if for all $\epsilon > 0$ there exists a sequence of intervals (a_n, b_n) (they need not be open intervals) such that

$$(1) \quad \sum b_n - a_n < \epsilon,$$
$$(2) \quad Z \subseteq \bigcup (a_n, b_n).$$

• **Theorem:** *If* f *is defined and bounded on a closed and bounded interval* $[a, b]$, *then* f *is Riemann integrable over* $[a, b]$ *if and only if* f *is continuous except, possibly, in a set of measure zero.*

• **Remarks:** We often say, if a function has a certain property (such as continuity, for example) except possibly in a set of measure zero, that the function has that property almost everywhere, abbreviated "a.e." We can thus say that a bounded function is Riemann integrable on a closed and bounded interval if and only if f is continuous a.e. in the interval.

• **More remarks:** A singleton is a set of measure zero. So is any countable set. But the Cantor set, which is uncountable, is also a set of measure zero. The union of a countable family of sets of measure zero is a set of measure zero. The union of an uncountable family of sets of measure zero is probably not a set of measure zero, tho it can be a set of measure zero. For example, the union of all the singleton subsets of $\mathbb{R}$ is all of $\mathbb{R}$, which is not a set of measure zero (a statement that requires proof!).

Sequences of functions, uniform convergence

Suppose that $\{f_n(x)\}_{n=0}^{\infty}$ is a sequence of real-valued functions, all defined on a domain D.

Definition: $f_n(x)$ *converges pointwise* to $f(x)$, defined on D, if for each $x \in D$, $f_n(x) \to f(x)$ as $n \to \infty$. The definition "in logic:" $(\forall x \in D)(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n \geq N \Longrightarrow |f_n(x) - f(x)| < \epsilon)$.

In words, a sequence of functions converges pointwise if for each x_o in the domain, the sequence of numbers given by $f_n(x_o)$ converges to the number $f(x_o)$, without any consideration of other possible values of x_o.

Definition: $f_n(x)$ *converges uniformly* on D to $f(x)$, defined on D, if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq N \Longrightarrow |f_n(x) - f(x)| < \epsilon$.

The definition "in logic:" $(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall x \in D)(\forall n \in \mathbb{N})(n \geq N \Longrightarrow |f_n(x) - f(x)| < \epsilon)$.
In words, a sequence of functions converges uniformly on D if for all $\epsilon > 0$ the graph of f_n is completely contained in the "ϵ-band" about the graph of f, namely the set of all points (x, y) such that $x \in D$ and $|y - f(x)| < \epsilon$.

How to check for uniform convergence

For each n, the graph of $f_n(x)$ just fits into the "closed" band of "height" $H_n := \sup_{x \in D} |f_n(x) - f(x)|$, that is it fits into the set $\{\, (x, y) : x \in D \wedge |y - f(x)| \leq H_n\}$. If we define $M_n := \sup_{x \in D} |f_n(x) - f(x)|$, then we have uniform convergence if and only if $M_n \to 0$ as $n \to \infty$.

In other words, the graphs of the f_n's get squeezed into smaller and smaller bands around the graph of the limit function f.

Example: $f_n(x) := x(1 - x^2)^n$ on $D = [-1, 1]$. For each *fixed* $x \in D$, $f_n(x) \to 0$. (Treat x=0 separately) Thus $f_n(x) \to 0$ pointwise.
Now we ask, does $f_n(x) \to 0$ uniformly? Method: find $M_n = \max f_n$. This takes some work: differentiate f_n, set it equal to zero, find roots. Ignore roots ± 1 since all f_n are 0 there. I get roots $\pm(2n+1)^{-1/2}$, so $M_n = (2n+1)^{-1/2}(2n/(2n+1))^n \to 0$ as $n \to \infty$. Hence the sequence converges uniformly to zero.

Another example: $f_n(x) := (1 - x^2)^n$. This does not converge uniformly; it does converge pointwise, to the function that is 0 except such that at 0, where it is 1. Proving non-uniform convergence can be done using theorems.

The most important theorem about uniform convergence for us is

Theorem: *If a sequence of continuous functions converges uniformly, then the limit function is continuous.*

Recall that the proof got from $f(x)$ to $f(x_o)$ in three steps.

Uniform convergence and the Riemann integral

Theorem: *If* $\{f_n(x)\}_{n=0}^{\infty}$ *is a sequence of real-valued functions that are Riemann integrable on a closed and bounded interval* $[a, b]$, *and the sequence converges uniformly to* $f(x)$ *on* $[a, b]$, *then* $f(x)$ *is Riemann integrable on* $[a, b]$ *and*

$$\lim_{n \to \infty} \int_a^b f_n(x)\, dx = \int_a^b \lim_{n \to \infty} f_n(x)\, dx = \int_a^b f(x)\, dx.$$

Remark: A question came up in class: How do you find the *value* of an integral defined this way? Here is one way: suppose that you know the function *is* Riemann integrable on $[a, b]$. Then you use this Theorem:

Theorem: *If* $f(x)$ *is Riemann integrable on a closed and bounded interval* $[a, b]$, *then* $\displaystyle\int_a^b f(x)\, dx = \lim_{N \to \infty} \mathcal{U}_{\Pi_N}$.

This means you need to know how to calculate $\mathcal{U}_{\Pi_N}$ (or you can use $\mathcal{L}_{\Pi_N}$, or even *Riemann sums,* R_π, which are defined over a partition $\pi \big| [a, b]$ by $R_\pi := \displaystyle\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$, where ξ_i can be any number in $[x_{i-1}, x_i]$).

This is what we have to do when we cannot use the Fundamental Theorem of Calculus, which is mostly restricted to continuous integrands.