Two comments: If $P(x)$ is a statement wih variable x, $[P(x)] := 1$ if $P(x)$ is true for x and $[P(x)] := 0$ if $P(x)$ is true for x (Iverson-Knuth notation: *Two Notes on Notation,* American Math. Monthly 99 (1992) 403 - 426). "Majorized by" means that the non-negative quantity to come is at least as large as the absolute vale, or norm, etc., of the quantity just presented.

Lemma: *For each positive integer n, and any function $c : \mathbb{N}^n \to \mathbb{C}$, there exists a function $f \in C_0^\infty$, with support in the closed unit ball in $\mathbb{R}^n$, such that $f^{(\beta)}(0) = c_\beta$, for all $\beta \in \mathbb{N}^n$.*

Proof: We begin with a function $\psi(x) \in C_0^\infty$ that depends only on $|x|$, is one if $|x| \le 1/2$, zero if $|x| \ge 1$, and decreases as $|x|$ increases.

(First condition on the ϵ_α) We will use a collection of positive numbers ϵ_α, all at most 1, that tends to zero as $|\alpha| \to \infty$, that will be restricted further as we proceed.

With the given conditions,

$$f(x) := \sum_\alpha c_\alpha \frac{x^\alpha}{\alpha!}\psi(x/\epsilon_\alpha)$$

converges for all x since the series is a finite sum for each $x \ne 0$ and $f(x) = 0$ if $|x| \ge 1$. In fact, if N is so large that $|\alpha| \ge N \Rightarrow \epsilon_\alpha < |x_o|/2$, all terms in the series with $|\alpha| \ge N$ are zero if $|x| \ge |x_o|/2$. Since the series is thus a finite sum in a neighborhood of each $x \ne 0$, f is C^∞ for $x \ne 0$.

To show that f is continuous at 0, we examine

$$f(x) - c_0 = \sum_{\alpha \ne 0} c_\alpha \frac{x^\alpha}{\alpha!}\psi(x/\epsilon_\alpha), \quad \text{majorized by} \quad \sum_{\alpha \ne 0} |c_\alpha| \frac{\epsilon_\alpha^{|\alpha|}}{\alpha!}\,[\,|x| < \epsilon_\alpha\,]$$

since $|x^\alpha \psi(x/\epsilon_\alpha)| \le \epsilon_\alpha^{|\alpha|}$ if $|x| < \epsilon_\alpha$ and $|x^\alpha\psi(x/\epsilon_\alpha)| = 0$ otherwise.

(Second condition on the ϵ_α) We assume that $\sum_\alpha |c_\alpha|\epsilon_\alpha < \infty$.

Because all the ϵ_α are at most one, $\sum_{\alpha\ne 0} |c_\alpha| \frac{\epsilon_\alpha^{|\alpha|}}{\alpha!} \le \sum_{\alpha \ne 0} |c_\alpha|\epsilon_\alpha < \infty$. By Dominated Convergence, f is continuous at zero. With

$$f_M(x) := \sum_{|\alpha| \le M} c_\alpha \frac{x^\alpha}{\alpha!}\psi(x/\epsilon_\alpha), \quad |f(x) - f_M(x)| \le \sum_{|\alpha| > M} |c_\alpha| \frac{\epsilon_\alpha^{|\alpha|}}{\alpha!} \to 0 \text{ uniformly.}$$

If we can show (by perhaps adding conditions on the ϵ_α) that $f^{(\beta)}(x) \to c_\beta$ as $x \to 0$, the Lemma will follow by inductions. We also would like to know that $(f - f_M)^{(\beta)} \to 0$ uniformly.

For $x \ne 0$

$$\begin{aligned} f^{(\beta)}(x) &= \sum_\alpha c_\alpha \left(\frac{\partial}{\partial x}\right)^\beta \left(\frac{x^\alpha}{\alpha!}\psi(x/\epsilon_\alpha)\right) = \sum_\alpha c_\alpha \sum_{\gamma \le \alpha,\,\beta} \frac{x^{\alpha-\gamma}}{(\alpha-\gamma)!}\psi^{(\beta-\gamma)}(x/\epsilon_\alpha)\epsilon_\alpha^{-|\beta-\gamma|} \\ &= c_\beta\psi(x/\epsilon_\beta) + c_\beta \sum_{\gamma<\beta} \frac{x^{\beta-\gamma}}{(\beta-\gamma)!}\psi^{(\beta-\gamma)}(x/\epsilon_\beta)\epsilon_\beta^{-|\beta-\gamma|} + \sum_{\alpha\ne\beta} c_\alpha \sum_{\gamma\le\alpha,\,\beta} \frac{x^{\alpha-\gamma}}{(\alpha-\gamma)!}\psi^{(\beta-\gamma)}(x/\epsilon_\alpha)\epsilon_\alpha^{-|\beta-\gamma|}. \end{aligned} \tag{1}$$

Next we consider (1) when $0 < |x| < \epsilon_\beta/2$. Then $\psi(x/\epsilon_\alpha) \equiv 1$ so the sum in the second term in the second line of (1) is zero since all the terms there with $\psi^{(\beta-\gamma)}(x/\epsilon_\beta)$ are zero. The first term is simply c_β.

(Third condition on the ϵ_α) We assume that $\epsilon_\alpha = \epsilon_\gamma$ if $|\alpha| = |\gamma|$ and $\epsilon_\alpha < \epsilon_\gamma/2$ if $|\alpha| > |\gamma|$.

Under the added conditions, in the double sum in the second line of (1) the terms with $\psi^{(\beta-\gamma)}(x/\epsilon_\alpha)$ are zero if $\gamma < \beta$ and $|\alpha| \le |\beta|$. Those terms with $\psi^{(\beta-\gamma)}(x/\epsilon_\alpha)$ and $\epsilon_\alpha \le |x|$ are also zero. The only terms that can "survive" require $|\alpha| > |\beta|$ and $|x| < \epsilon_\alpha$, with $\gamma \le \beta$. Therefore

$$f^{(\beta)}(x) - c_\beta = \sum_{|\alpha|>|\beta|} c_\alpha \sum_{\gamma\le\alpha,\,\beta} \frac{x^{\alpha-\gamma}}{(\alpha-\gamma)!}\psi^{(\beta-\gamma)}(x/\epsilon_\alpha)\epsilon_\alpha^{-|\beta-\gamma|}. \tag{2}$$

Since here $|\alpha-\gamma| = |\alpha| - |\gamma|$ (same with β) the difference is majorized by

$$\sum_{|\alpha|>|\beta|} |c_\alpha| \sum_{\gamma\le\alpha,\,\beta} \frac{|x|^{|\alpha|-|\beta|}}{(\alpha-\gamma)!} |\psi^{(\beta-\gamma)}(x/\epsilon_\alpha)| (|x|/\epsilon_\alpha)^{|\beta-\gamma|}; \tag{3}$$

The exponent on $|x|$ is at least one and $|x|<1$ (for each α with $|\alpha|>|\beta|$ we need $|x|<\epsilon_\alpha$, else the terms do not survive) so (3) is in turn majorized by

$$\sum_{|\alpha|>|\beta|} |c_\alpha| \min\{|x|,\,\epsilon_\alpha\} \sum_{\gamma\le\beta} |\psi^{(\beta-\gamma)}(x/\epsilon_\alpha)| (|x|/\epsilon_\alpha)^{|\beta-\gamma|} \le K_\beta \sum_{|\alpha|>|\beta|} |c_\alpha| \min\{|x|,\,\epsilon_\alpha\}, \tag{4}$$

where

$$K_\beta := \sum_{\gamma\le\beta} \max_{|z|\le 1} |\psi^{(\beta-\gamma)}(z)| |z|^{|\beta-\gamma|},$$

a constant that depends only on β and ψ. By Dominated Convergence, $f^{(\beta)}(x)\to c_\beta$ as $x\to 0$. For each β with $|\beta|=1$ this implies that $f^{(\beta)}(0)=c_\beta$ and that $f^{(\beta)}$ is continuous. By inductions, the Lemma follows.

Now we suppose that $M>|\beta|$. Then

$$\begin{aligned} f_M^{(\beta)}(x) &= \sum_{|\alpha|\le M} c_\alpha \left(\frac{\partial}{\partial x}\right)^\beta \left(\frac{x^\alpha}{\alpha!}\psi(x/\epsilon_\alpha)\right) = \sum_{|\alpha|\le M} c_\alpha \sum_{\gamma\le\alpha,\,\beta} \frac{x^{\alpha-\gamma}}{(\alpha-\gamma)!} \psi^{(\beta-\gamma)}(x/\epsilon_\alpha)\epsilon_\alpha^{-|\beta-\gamma|} \\ &= c_\beta\psi(x/\epsilon_\beta) + c_\beta \sum_{\gamma<\beta} \frac{x^{\beta-\gamma}}{(\beta-\gamma)!}\psi^{(\beta-\gamma)}(x/\epsilon_\beta)\epsilon_\beta^{-|\beta-\gamma|} + \sum_{\alpha\ne\beta}^{*} c_\alpha \sum_{\gamma\le\alpha,\,\beta}^{*} \frac{x^{\alpha-\gamma}}{(\alpha-\gamma)!}\psi^{(\beta-\gamma)}(x/\epsilon_\alpha)\epsilon_\alpha^{-|\beta-\gamma|}, \end{aligned} \tag{5}$$

it being understood that in the starred sums the restrictions that $M\ge|\alpha|$ and $M>|\beta|$ hold. The restriction that $M>|\beta|$ allows α to be equal to β. We consider (1) and (5) now for all x, not just those x with $0<|x|<\epsilon_\beta/2$. Only the double sums appear in the difference. Then

$$f^{(\beta)}(x) - f_M^{(\beta)}(x) = \sum_{|\alpha|>M} c_\alpha \sum_{\gamma\le\alpha,\,\beta} \frac{x^{\alpha-\gamma}}{(\alpha-\gamma)!}\psi^{(\beta-\gamma)}(x/\epsilon_\alpha)\epsilon_\alpha^{-|\beta-\gamma|},$$

majorized by

$$\sum_{|\alpha|>M} |c_\alpha| \sum_{\gamma\le\alpha,\,\beta} \frac{|x|^{|\alpha-\gamma|}}{(\alpha-\gamma)!} \left|\psi^{(\beta-\gamma)}(x/\epsilon_\alpha)\right| (|x|/\epsilon_\alpha)^{|\beta-\gamma|} \le \sum_{|\alpha|>M} |c_\alpha|\epsilon_\alpha \cdot K_\beta$$

since, *unless* $|x|<\epsilon_\alpha<1$, $\left|\psi^{(\beta-\gamma)}(x/\epsilon_\alpha)\right| = 0$. Hence $f^{(\beta)} - f_M^{(\beta)} \to 0$ uniformly as $M\to\infty$.

From (1),

$$f^{(\beta)}(x) = \sum_\delta c_\delta \sum_{\gamma\le\delta,\,\beta} \frac{x^{\delta-\gamma}}{(\delta-\gamma)!}\psi^{(\beta-\gamma)}(x/\epsilon_\delta)\epsilon_\delta^{-|\beta-\gamma|}.$$