Collected here: various details that would take up too much class time.

7/1/2005: We wanted to show that when $f(x)$ had an extreme value at an *interior point* X of an interval $[a,\, b]$ (i.e., $X \in (a,\, b)$) and $f'(X)$ exists, then $f'(X) = 0$. See also §**1.2** and after Theorem III in §**1.12**.

Proof: This uses the following result that we can call a Lemma:

Lemma: *If h is defined on an interval I and x_o is either in I or x_o is an endpoint of I and $\lim\limits_{\substack{x\to x_o\\ x\in I}} h(x)$ exists and $h(x) \geq 0$ in I when $x \neq x_o$, then $\lim\limits_{\substack{x\to x_o\\ x\in I}} h(x) \geq 0$.*

Let's use the lemma first, then prove it. We chose to deal with the case when $f(X)$ is an interior *maximum* value of the function f on $[a,\, b]$. We want to show that

$$f'(X) = \lim_{x\to X} \frac{f(t)-f(X)}{t-X} = 0.$$

We work with the *difference quotient* $\dfrac{f(t)-f(X)}{t-X}$ on opposite sides of X. If $a \leq t < X$ then the numerator in the difference quotient is less than or equal to zero and the denominator is negative.

Thus the function $h(t) := \dfrac{f(t)-f(X)}{t-X} \geq 0$ in the interval $I := [a,\, X)$ and $\lim\limits_{\substack{t\to X\\ t\in I}} h(t) = f'(X)$ exists. Therefore by the Lemma $f'(X) \geq 0$. On the other side of X, in the interval $I := (X,\, b]$, $h(t) := -\dfrac{f(t)-f(X)}{t-X} \geq 0$ because now the numerator in the difference quotient is less than or equal to zero and the denominator is positive. We then multiply by -1 to make $h(t) \geq 0$ in I. By the Lemma, $-f'(X) \geq 0$. By Trichotomy $f'(X) = 0$.

Proof of the Lemma

We let $L := \lim\limits_{\substack{x\to x_o\\ x\in I}} h(x)$. We want to prove that $L \geq 0$. By the hypothesis that the limit of $h(x)$ exists, for every $\epsilon > 0$ there exists $\delta > 0$ such that $x \in I$ and $0 < |x - x_o| < \delta$ implies that $|h(x) - L| < \epsilon$. Then for $x \in I$ and $0 < |x - x_o| < \delta$,

$$0 \leq h(x) = (h(x) - L) + L \leq |h(x) - L| + L < \epsilon + L.$$

But then $0 \leq \epsilon + L$ for every $\epsilon > 0$. To conclude that $L \geq 0$ we assume not, namely that $L < 0$. Then we choose $\epsilon := -L/2 > 0$, and so we have proved that

$$0 \leq -(L/2) + L = L/2 < 0, \quad \text{which is absurd. The proof is done.}$$

(the phrase "which is absurd" means "which is a contradiction")