An Example

To begin, suppose we are given the differential equation $Ly = y'' + 2y' - 3y = 0$. The standard procedure is to insert $y_r = e^{rx}$ into the left-hand-side of the equation, which gives us

$$Ly_r = y_r'' + 2y_r' - 3y_r = r^2e^{rx} + 2re^{rx} - 3e^{rx} = (r^2 + 2r - 3)e^{rx}, \quad \text{and this is zero if and only if} \quad r^2 + 2r - 3 = 0.$$

Thus the functions e^x and e^{-3x} are solutions of the equation $Ly = 0$, as well as all their linear combinations, $ae^x + be^{-3x}$, no matter what scalars a and b we choose. In symbols, $\text{span}\{e^x, e^{-3x}\} \subseteq \text{Ker}L$.

In a TA session, someone raised the question: "How do we know that these are *all* the solutions?" That is, why is it true that, also, $\text{Ker}L \subseteq \text{span}\{e^x, e^{-3x}\}$?

Our task, then, is to start with a solution y, given us by an "opponent" who is obliged to give us answers to specific questions about values of that function y, and its derivatives, but who does not have to answer any other questions, such as "Is y in $\text{span}\{e^x, e^{-3x}\}$?" We seek to "win" by answering the question ourselves.

We start by asking for $y(0)$ and $y'(0)$. Let's suppose we are told that $y(0) = 5$ and that $y'(0) = 1$. We return to our laboratory and try to construct a function

$$u = ae^x + be^{-3x} \quad \text{such that} \quad u(0) = 5 \quad \text{and} \quad u'(0) = 1.$$

Thinking of a and b as unknowns we have $u(0) = a + b$ and $u'(0) = a - 3b$, so we seek a and b so that $a + b = 5$ and $a - 3b = 1$. This system of two equations in two unknowns has the unique solution $a = 4$, $b = 1$, so we define $u(x) := 4e^x + e^{-3x}$. We also know that $Lu \equiv 0$.

Now we know that if we define $v := y - u$ then $Lv \equiv 0$, and that $v(0) = 0 = v'(0)$.

Suppose we knew that the only v such that $Lv \equiv 0$ and $v(0) = 0 = v'(0)$ is the function $v \equiv 0$. Then we would know that $y \equiv u$. Since $u \in \text{span}\{e^x, e^{-3x}\}$, y would belong to $\text{span}\{e^x, e^{-3x}\}$, so $\text{Ker}L \subseteq \text{span}\{e^x, e^{-3x}\}$ would indeed be true. So now we seek to verify this.

We will use a trick to convert a second-order differential equation into a first-order differential equation for a vector-valued function, a tool called Gronwall's Lemma, and (more than once) the Schwarz Inequality.

The conversion trick: We begin with $Lv = v'' + 2v' - 3v = 0$. We then set $v_1 := v$ and $v_2 = v'$ and so define the vector-valued function $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. Then $\mathbf{v}(0) = \begin{pmatrix} v(0) \\ v'(0) \end{pmatrix} = \mathbf{0}$ and (since $v'' = -2v' + 3v$)

$$\mathbf{v}' = \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} v' \\ v'' \end{pmatrix} = \begin{pmatrix} v_2 \\ -2v' + 3v \end{pmatrix} = \begin{pmatrix} v_2 \\ -2v_2 + 3v_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix} \mathbf{v}.$$

Since integrating a vector-valued function amounts to integrating the component functions, this last equation can be integrated and we get

$$\text{(1)} \qquad \mathbf{v}(x) = \mathbf{v}(0) + \int_0^x \mathbf{v}'(t)\,dt = \mathbf{0} + \int_0^x A\mathbf{v}(t)\,dt = \int_0^x A\mathbf{v}(t)\,dt = A\int_0^x \mathbf{v}(t)\,dt,$$

where $A = \begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix}$. We were able to factor out the matrix A because it consists of constants and because an integral is a "sum."

To see where we're heading, let's see what Gronwall's Lemma says.

Gronwall's Lemma: *Suppose that* $f(t) \geq 0$ *on an interval* $[a, b]$, *that* f *is continuous on* $[a, b]$ *and that* $f(x) \leq C\int_a^x f(t)\,dt$ *for some positive constant* C *and for all* $x \in [a, b]$. *Then* $f(x) \equiv 0$ *on* $[a, b]$.

There is a bit of resemblence between (1) and the inequality in Gronwall's Lemma. If we could replace $\mathbf{v}(t)$ by $f(t) := |\mathbf{v}(t)|$ and replace the matrix A by a constant C, and replace $=$ by $\leq$ in (1), we would have the desired result! Of course we'll have to check that $|\mathbf{v}(t)|$ is a continuous function of t. We'll do that later on.

First we note that $|\mathbf{v}(x)| = \left|A\int_0^x \mathbf{v}(t)\,dt\right|$. Thus the left side is taken care of. We have to get rid of the A. Somehow we need to get the function $|\mathbf{v}(t)|$ on the right.

If we think of the vector $\int_0^x \mathbf{v}(t)\,dt$ as just a vector $\mathbf{x}$, we have $\left|A\int_0^x \mathbf{v}(t)\,dt\right| = |A\mathbf{x}|$. We want to replace $|A\mathbf{x}|$ by $C|\mathbf{x}|$. This is one of the places we will have to "pay" for the desired simplification by giving up equality, and settling instead for the *inequality* $|A\mathbf{x}| \le C|\mathbf{x}|$.

As is usual we work with squares:

$$|A\mathbf{x}|^2 = \left|\begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix}\mathbf{x}\right|^2 = \left|\begin{pmatrix} x_2 \\ -2x_2 + 3x_1 \end{pmatrix}\right|^2 = x_2^2 + (3x_1 - 2x_2)^2 \le x_2^2 + (3^2 + (-2)^2)(x_1^2 + x_2^2),$$

where the inequality is by Schwarz's Inequality (with both sides squared). Simplifying gives $|A\mathbf{x}|^2 \le 13x_1^2 + 14x_2^2 \le 14|\mathbf{x}|^2$. Again we paid for simplicity by making the inequality worse. But we accomplished one thing: we got the A and the $\mathbf{x}$ separated! Thus so far we have (replacing our nickname $\mathbf{x}$ by what we had):

$$(2) \qquad |\mathbf{v}(x)| \le \sqrt{14}\left|\int_0^x \mathbf{v}(t)\,dt\right|, \quad \text{and we want} \quad |\mathbf{v}(x)| \le \sqrt{14}\int_0^x |\mathbf{v}(t)|\,dt \quad \text{instead.}$$

To use Schwarz's Inequality we let $\mathbf{w}$ denote an arbitrary vector (in $\mathbb{R}^2$) and we get (because we integrate each component)

$$\int_0^x \mathbf{v}(t)\,dt \bullet \mathbf{w} = \int_0^x \mathbf{v}_1(t)\,dt\,w_1 + \int_0^x \mathbf{v}_2(t)\,dt\,w_2 = \int_0^x (\mathbf{v}_1(t)w_1 + \mathbf{v}_2(t)w_2)\,dt = \int_0^x \mathbf{v}(t) \bullet \mathbf{w}\,dt.$$

By the triangle inequality for integrals we have

$$\left|\int_0^x \mathbf{v}(t)\,dt \bullet \mathbf{w}\right| = \left|\int_0^x \mathbf{v}(t) \bullet \mathbf{w}\,dt\right|. \le \int_0^x |\mathbf{v}(t) \bullet \mathbf{w}|\,dt \le \int_0^x |\mathbf{v}(t)|\,|\mathbf{w}|\,dt \le \int_0^x |\mathbf{v}(t)|\,dt\,|\mathbf{w}|, \quad \text{namely}$$

$$\left|\int_0^x \mathbf{v}(t)\,dt \bullet \mathbf{w}\right| \le \int_0^x |\mathbf{v}(t)|\,dt\,|\mathbf{w}|,$$

by the Schwarz Inequality, used under the integral sign, and then we factor out $|\mathbf{w}|$. This is valid for every vector $\mathbf{w} \in \mathbb{R}^2$. We now do something clever, and put $\mathbf{w} = \int_0^x \mathbf{v}(t)\,dt$, which gives, on substitution,

$$\left|\int_0^x \mathbf{v}(t)\,dt\right|^2 = \left|\int_0^x \mathbf{v}(t)\,dt \bullet \int_0^x \mathbf{v}(t)\,dt\right| \le \int_0^x |\mathbf{v}(t)|\,dt \cdot \left|\int_0^x \mathbf{v}(t)\,dt\right|.$$

If $\left|\int_0^x \mathbf{v}(t)\,dt\right| \ne 0$ we can cancel on both sides; if $\left|\int_0^x \mathbf{v}(t)\,dt\right| = 0$ we can't cancel but it is still true that

$$\left|\int_0^x \mathbf{v}(t)\,dt\right| \le \int_0^x |\mathbf{v}(t)|\,dt$$

because then the left-hand-side would be zero anyway, and the right-hand-side non-negative. Thus what we wanted, the last part of (2), is true.

By Gronwall's Lemma, $|\mathbf{v}(x)| \equiv 0$, as desired.

This means that $\mathbf{v}(x) \equiv 0$, so in particular the first component, $v_1(x) \equiv 0$. Since $v_1(x) = y(x) - u(x)$, we have shown that $y(x) = u(x)$ for all x, and hence that all along, y was in $\operatorname{span}\{e^x, e^{-3x}\}$. In fact we have shown that $y(x) = 4e^x + e^{-3x}$.

To show that *every* solution of $y'' + 2y' - 3y = 0$ is a linear combination of e^x and e^{-3x} we ask our opponent for $y(0)$ and $y'(0)$, then we seek

$$u = ae^x + be^{-3x} \quad \text{such that} \quad u(0) = y(0) \text{ and } u'(0) = y'(0).$$

Thinking of a and b as unknowns we have $u(0) = a + b$ and $u'(0) = a - 3b$, so we seek a and b so that $a + b = y(0)$ and $a - 3b = y'(0)$. Indeed, $b = (y(0) - y'(0))/4$ by subtracting the second equation from the first, so that $a = y(0) - b = (3y(0) + y'(0))/4$.

We can then continue exactly as we did before, and now we can conclude that every solution of $y'' + 2y' - 3y = 0$ is a linear combination of e^x and e^{-3x}.

Now let us prove Gronwall's Lemma. Later we will fill in some details we skipped, because some of you will catch them!

Proof of Gronwall's Lemma

We will need to use the fact that a continuous function on a closed and bounded interval $[a, b]$ always takes on maximum and minimum values. That is, if $f(x)$ is continuous on $[a, b]$ then there exist x_m and x_m, both in $[a, b]$, such that $f(x_m) \leq f(x) \leq f(x_M)$ for all x in $[a, b]$.

Let us choose $x_o := a + \min\{\frac{1}{2C}, b - a\}$. Then $a < x_o \leq b$. Since f is continuous in $[a, b]$, f is continuous in $[a, x_o]$. We choose x_M in $[a, x_o]$ such that $f(x) \leq f(x_M)$ for all x in $[a, x_o]$. By hypothesis

$$0 \leq f(x_M) \leq C \int_a^{x_M} f(t)\,dt \leq C \int_a^{x_o} f(t)\,dt \leq C \int_a^{x_o} f(x_M)\,dt = Cf(x_M) \int_a^{x_o} dt = C(x_o - a)f(x_M).$$

But by our clever choice of C we know that $x_o - a \leq \frac{1}{2C}$ so

$$0 \leq f(x_M) \leq C \cdot \frac{1}{2C} \cdot f(x_M) = f(x_M)/2.$$

This means that $f(x_M) = 0$. Since $0 \leq f(x) \leq f(x_M)$ in $[a, x_o]$, $f(x) \equiv 0$ in $[a, x_o]$. If $x_o < b$ we can start over, replacing $[a, b]$ by $[x_o, b]$ and repeat the argument, or simply apply what we have already done to the new interval. We can thus step-by-step enlarge the interval on which $f(x) \equiv 0$ until it covers the whole interval. This completes the proof of Gronwall's Lemma.

Notes and Remarks

Distinct roots

What we did for the equation $y'' + 2y' - 3y = 0$ we can do for any equation $y'' + ay' + by = 0$ as long as the roots of $r^2 + ar + b = 0$ are distinct, that is, if $r^2 + ar + b = (r - r_1)(r - r_2)$ and $r_1 \neq r_2$. All we have to do in the foregoing work is to replace e^x and e^{-3x} by $e^{r_1 x}$ and $e^{r_2 x}$, then use $y(0)$ and $y'(0)$ instead of 5 and 1. This works even if r_1 and r_2 are complex roots. In that case we can make use of Euler's(?) Formula

$$e^{s+it} = e^s(\cos t + i \sin t)$$

or simply work with e^z is the "usual" way, which lets us say that $\frac{d}{dx} e^{zx} = ze^{zx}$.

Double roots

If the roots are not distinct, say if we work with $y'' + 4y' + 4y = 0$, our auxilary polynomial becomes $r^2 + 2r + 4 = (r + 2)^2$, which gives us two roots equal to -2. This gives us only one function, e^{-2x}, to use. We cannot construct a solution that has arbitrary assigned values of function and derivative at $x = 0$. We need another solution.

If we apply $L := D^2 + 4D + 4I$ to e^{rx} we get $L(e^{rx}) = (r + 2)^2 e^{rx}$. Now we do something clever: we differentiate with respect to r, holding x constant. This is usually expressed as $\frac{\partial}{\partial r}$:

$$\frac{\partial}{\partial r} L(e^{rx}) = \frac{\partial}{\partial r} \left[(r + 2)^2 e^{rx}\right] = 2(r + 2)e^{rx} + (r + 2)^2 re^{rx} = [2(r + 2) + (r + 2)^2 r]e^{rx}. \tag{3}$$

Since x and r are independent and since e^t has continuous derivatives of all orders(!), we can use the Chain Rule, and interchange the order of differentiating with respect to r and x. When we combine this with (3) we get

$$\frac{\partial}{\partial r} L(e^{rx}) = L(\frac{\partial}{\partial r} e^{rx}) = L(xe^{rx}) = [2(r + 2) + (r + 2)^2 r]e^{rx}.$$

When $r = -2$, we have shown that $L(xe^{-2x}) \equiv 0$. This gives us the other solution we need. The set $\{e^{-2x}, xe^{-2x}\}$ is linearly independent set of functions. If we define $u(x) := ae^{-2x} + bxe^{-2x}$ we can still solve $u(0) = y(0)$ and $u'(0) = y'(0)$ for the unknowns a and b, set $v := y - u$ and proceed as before. Thus we can handle *every* equation $y'' + 2y' - 3y = 0$.

Higher order equations

We can use exactly the same ideas to handle equations such as $y^{(4)} + 3y^{(3)} - y'' + 5y = 0$: convert to a first-order equation for a vector-valued function with values in $\mathbb{R}^4$. We just let $y_1 = y$, $y_2 = y'$, $y_3 = y''$, $y_3 = y^{(3)}$. The same ideas work, with more technical complications. We need one more tool. The Fundamental Theorem of Algebra, which asserts that every polynomial of degree n, having complex coeffieients (that includes real coefficients!), has exactly n complex roots (that might all be real!). Some of the roots might be repeated. If the distinct roots are $r_1, r_1, \ldots, r_k$, with respective *multiplicities* $m_1, m_1, \ldots, m_k$, then $m_1 + m_1 + \cdots + m_k = n$. Thus the auxilary polynomial for our fourth-order example, $y^{(4)} + 3y^{(3)} - y'' + 5y = 0$, which is $r^4 + 3r^3 - r^2 + 5$, has four roots. They might all be the same, or all different, or in between. *Finding* the roots in this example is possible, in principle, but it might be *very* hard. Polynomials of degree higher than four need not be solvable at all by algebraic methods. Nevertheless they "exist." We can build solutions in the same way, but multiplicities have to be taken into account. If r_1 has multiplicity $m_1 > 1$, the functions $e^{r_1x}, xe^{r_1x}, x^2e^{r_1x}, \ldots, x^{m_1-1}e^{r_1x}$ serve as linearly independent solutions corresponding to the root r_1. Roots r that occur singly contribute only e^{rx} to the list of (literally) basic solutions. All together we get n functions for an n-th order equation.

The inequality we got that led to our use of Gronwall's Lemma is done the same way; it just requires a longer calculation. Instead of the $\sqrt{14}$ we got in our original example, we'll get $\sqrt{1 + 9 + 1 + 25} = \sqrt{36} = 6$ as our constant C.

Thus while we may not be able to carry out the details, we "know" that every solution of a homogeneous linear differential equation with constant coefficients is in the span of a set of n linearly independent functions. In other words, we have identified the nullspace of every homogeneous linear differential operator with constant coefficients.

Since we can't solve every polynomial equation, mathematicians have had to ask how well knowing *approximate* roots works in the context of linear differential equations with constant coefficients.

The continuity of $|\mathbf{v}(t)|$

We know from the differential equation that v'' exists at each point, and therefore that v' is continuous. And since v' exists at each point, v is also vontinuous. This means that $\mathbf{v}(t)$ is continuous, because each of its component functions is continuous. In particular this means that for each t_o, $\lim_{t \to t_o} \mathbf{v}(t) = \mathbf{v}(t_o)$, or that $\lim_{t \to t_o} |\mathbf{v}(t) - \mathbf{v}(t_o)| = 0$. What we need to do is check that $\lim_{t \to t_o} ||\mathbf{v}(t)| - |\mathbf{v}(t_o)|| = 0$. It will be enough to show that

$$||\mathbf{v}(t)| - |\mathbf{v}(t_o)|| \leq |\mathbf{v}(t) - \mathbf{v}(t_o)| \text{ since } \lim_{t \to t_o} |\mathbf{v}(t) - \mathbf{v}(t_o)| = 0.$$

This is an application of the Triangle Inequality; it has nothing to do with differential equations as such.

Lemma: *For all vectors* $\mathbf{x}$ *and* $\mathbf{y}$ *in any* $\mathbb{R}^n$, $||\mathbf{x}| - |\mathbf{y}|| \leq |\mathbf{x} - \mathbf{y}|$.

Proof: We do this in two nearly identical steps. First, $|x| = |x - y + y| \leq |x - y| + |y|$ by the Triangle Inequality, so $|x| - |y| \leq |x - y|$. Letting x and y switch roles we have $|y| = |y - x + x| \leq |y - x| + |x| = |x - y| + |x|$, so $|y| - |x| \leq |x - y|$. Since the bigger of $|x| - |y|$ and $|y| - |x|$ is $||x| - |y||$, we have shown that $||x| - |y|| \leq |x - y|$.

Conclusion

In this note we have shown that the nullspace of a linear differential operator L with constant coefficients is the span of the basic solutions of $Ly = 0$ that are derived from the roots of the corresponding auxilary polynomial, with multiciplities taken into account. We used as tools the Schwarz Inequality, the Triangle Inequality, Gronwall's Lemma, the maximum-minimum theorem and the Fundamental Theorem of Algebra. We also used a trick to convert a higher-order differential equation into a first-order vector-valued differential equation and the idea to simplify expressions by using inequalities.