

(1) **Theorem:** If a power series  $\sum_{n=0}^{\infty} c_n z^n$  converges and  $z_o \neq 0$  then the series  $\sum_{n=0}^{\infty} c_n z^n$  converges absolutely for all  $z$  with  $|z| < |z_o|$  and the function  $f(z)$  defined by  $f(z) := \sum_{n=0}^{\infty} c_n z^n$  is analytic for all  $z$  with  $|z| < |z_o|$ .

Before we begin the proof, let us list some facts about series and some examples we will use. Much of what we use here can be found in Chapter 5. I'll try to say what the idea is behind each step.

(2) If a sequence  $\{a_n\}$  of complex numbers converges, then  $|a_n| \leq M$  for some  $M < \infty$ .

(3) If a series  $\sum_{n=0}^{\infty} a_n$  converges, then its terms  $a_n$  converge to zero.

(4) If a series  $\sum_{n=0}^{\infty} a_n$  converges absolutely, then it converges.

(5) If a series  $\sum_{n=0}^{\infty} a_n$  converges absolutely and  $|b_n| \leq K|a_n|$  for all  $n$ , then  $\sum_{n=0}^{\infty} b_n$  converges absolutely.

(6) The series  $\sum_{n=1}^{\infty} n^m z^n$  converges absolutely for  $|z| < 1$  for all integers  $m$ .

(7) If  $A$  and  $B$  are complex numbers and  $n$  is a positive integer then  $A^n - B^n = (A - B) \sum_{k=0}^{n-1} A^{n-1-k} B^k$ .

*Proof of (1):* First we prove the convergence for  $|z| < |z_o|$ . Since the series  $\sum_{n=0}^{\infty} c_n z_o^n$  converges its terms,  $c_n z_o^n$ , converge to zero and hence are bounded, by (3) and then (2), i.e.,  $|c_n z_o^n| \leq M$  for some finite  $M$ . We set  $\rho := (|z| + |z_o|)/2$ . Then

$$c_n z^n = c_n z_o^n \frac{z^n}{z_o^n} \text{ so } |c_n z^n| = |c_n z_o^n| \left(\frac{|z|}{|z_o|}\right)^n \leq M \left(\frac{|z|}{|z_o|}\right)^n < M \left(\frac{\rho}{|z_o|}\right)^n.$$

Since  $\frac{\rho}{|z_o|} < 1$  we now use (6) (with  $m = 0$ ) and (5) to complete the convergence proof.

The idea behind the proof of differentiability for each  $z$  with  $0 \leq |z| < |z_o|$  is term-by-term differentiation. That is, we want to show that

$$(8) \quad \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = \sum_{n=1}^{\infty} n c_n z^{n-1}.$$

Note that we need to prove that the series on the right converges when  $|z| < |z_o|$  as well as prove the limit of the difference quotient exists. We begin by writing the difference quotient as a series. We will work only with those nonzero  $h$  that satisfy  $|z| + |h| < \rho$ . This ensures that the series converges for  $z+h$  as well as for  $z$ . The difference quotient becomes

$$\frac{f(z+h) - f(z)}{h} = \sum_{n=0}^{\infty} c_n \frac{(z+h)^n - z^n}{h} = \sum_{n=1}^{\infty} c_n \sum_{k=0}^{n-1} (z+h)^{n-1-k} z^k \text{ by using (7) on each term. The } h \text{ cancels.}$$

The reason there is no  $n = 0$  term is that its numerator is zero. When  $n = 1$  the inner sum has only one term, which is 1, so the term for  $n = 1$  is just  $c_1$ . We can "clean up" the sum and use a second idea – write down the difference of the two series in (8) and show that the difference tends to zero as  $h \rightarrow 0$ :

$$\frac{f(z+h) - f(z)}{h} - \sum_{n=1}^{\infty} n c_n z^{n-1} = c_1 + \sum_{n=2}^{\infty} c_n \sum_{k=0}^{n-1} (z+h)^{n-1-k} z^k - \sum_{n=1}^{\infty} n c_n z^{n-1}.$$

We note that the first term in the second series is also  $c_1$ , so those terms cancel and we combine the series:

$$(9) \quad \frac{f(z+h) - f(z)}{h} - \sum_{n=1}^{\infty} n c_n z^{n-1} = \sum_{n=2}^{\infty} \left[ \left( c_n \sum_{k=0}^{n-1} (z+h)^{n-1-k} z^k \right) - n c_n z^{n-1} \right].$$

We now use a little algebraic trickery:  $n c_n z^{n-1} = c_n \sum_{k=0}^{n-1} z^{n-1} = c_n \sum_{k=0}^{n-1} z^{n-1-k} z^k$ . We put this sum into (9) in place of  $n c_n z^{n-1}$  and factor out  $c_n$ , obtaining

$$\frac{f(z+h) - f(z)}{h} - \sum_{n=1}^{\infty} n c_n z^{n-1} = \sum_{n=2}^{\infty} c_n \sum_{k=0}^{n-1} \{ (z+h)^{n-1-k} - z^{n-1-k} \} z^k.$$

Once again we have a difference of powers, so we use (7) again and arrive at a messy expression:

$$(10) \quad \frac{f(z+h) - f(z)}{h} - \sum_{n=1}^{\infty} n c_n z^{n-1} = \sum_{n=2}^{\infty} c_n \sum_{k=0}^{n-1} \left( h \sum_{\ell=0}^{n-2-k-\ell} (z+h)^{n-2-k-\ell} z^\ell \right) z^k = \sum_{n=2}^{\infty} c_n h S(z, h),$$

where

$$S(z, h) = \sum_{k=0}^{n-1} \left( \sum_{\ell=0}^{n-2-k-\ell} (z+h)^{n-2-k-\ell} z^\ell \right) z^k.$$

The next idea here is to find a simple estimate for  $|S(z, h)|$  that we can use to estimate the terms in  $\sum_{n=2}^{\infty} c_n h S(z, h)$  so that we can apply the Comparison Test, (5). The idea for the estimate is to replace everything by its absolute value and then notice that  $|z| < \rho$  and  $|z+h| < \rho$ . This gives us (using the triangle inequality at two levels)

$$|S(z, h)| \leq \sum_{k=0}^{n-1} \sum_{\ell=0}^{n-2-k-\ell} |z+h|^{n-2-k-\ell} |z|^\ell |z|^k < \sum_{k=0}^{n-1} \sum_{\ell=0}^{n-2-k-\ell} \rho^{n-2-k-\ell} \rho^\ell \rho^k = \sum_{k=0}^{n-1} \sum_{\ell=0}^{n-2-k-\ell} \rho^{n-2}.$$

Obtaining our estimate now becomes a matter of counting the number of  $\rho^{n-2}$ 's in the last double sum. But let's be lazy and notice that that double sum is less than  $\sum_{k=0}^{n-1} \sum_{\ell=0}^{n-2-k-\ell} \rho^{n-2} = n(n-1)\rho^{n-2} < n^2 \rho^{n-2}$ . We can put this estimate into (10) and use the triangle inequality (it works for series) to get

$$(11) \quad \left| \frac{f(z+h) - f(z)}{h} - \sum_{n=1}^{\infty} n c_n z^{n-1} \right| \leq \sum_{n=2}^{\infty} |c_n| |h| |S(z, h)| < \sum_{n=2}^{\infty} |c_n| |h| n^2 \rho^{n-2} = |h| \sum_{n=2}^{\infty} n^2 |c_n| \rho^{n-2}.$$

Here the last quantity on the right is  $|h|$  times either a number or  $+\infty$ . If we can show that the series  $\sum_{n=2}^{\infty} n^2 |c_n| \rho^{n-2}$  converges, we will have shown that  $\frac{f(z+h) - f(z)}{h} - \sum_{n=1}^{\infty} n c_n z^{n-1} \rightarrow 0$  as  $h \rightarrow 0$ . The series has  $|c_n|$  in it so we have to use our original estimate  $|c_n z_o^n| \leq M$ . We then have

$$\sum_{n=2}^{\infty} n^2 |c_n| \rho^{n-2} = \rho^{-2} \sum_{n=2}^{\infty} n^2 |c_n| \rho^n = \rho^{-2} \sum_{n=2}^{\infty} n^2 |c_n z_o^n| \left( \frac{\rho}{|z_o|} \right)^n \leq M \rho^{-2} \sum_{n=2}^{\infty} n^2 \left( \frac{\rho}{|z_o|} \right)^n.$$

We can now use (6) (with  $m = 2$ ) to complete the verification that  $\sum_{n=2}^{\infty} n^2 |c_n| \rho^{n-2}$  converges. This completes the proof that  $f'(z)$  exists for every  $z$  with  $|z| < |z_o|$ .

**Remark:** We could have stated the Theorem, (1), in a slightly different way:

(12) **Theorem:** If a power series  $\sum_{n=0}^{\infty} c_n z_o^n$  converges and  $z_o \neq 0$  then the series  $\sum_{n=0}^{\infty} c_n z^n$  converges absolutely for all  $z$  with  $|z| < |z_o|$  and the function  $f(z)$  defined by  $f(z) := \sum_{n=0}^{\infty} c_n z^n$  is differentiable in the complex sense for all  $z$  with  $|z| < |z_o|$ , and

$$f'(z) = \sum_{n=1}^{\infty} n c_n z^{n-1} \text{ for all } z \text{ with } |z| < |z_o|.$$

**Example:**  $\sum_{n=1}^{\infty} \frac{z_o^n}{n}$  converges when  $z_o = -1$ , though not absolutely.