We have

$$F(\xi,\,x) = e^{-(x+\frac{i\xi}{2})^2} = e^{-x^2}\,e^{-ix\xi}\,e^{\frac{\xi^2}{4}}$$

and we want to show that, for each ξ, there is an integrable function $g(x)$ such that, for all η sufficiently close to zero (we'll assume that $|\eta| \le 1$),

$$\left|\frac{F(\xi+\eta,\,x) - F(\xi,\,x)}{\eta}\right| \le g(x).$$

We know from the formula that the limit (as $\eta \to 0$) exists. We start with

$$F(\xi+\eta,\,x) - F(\xi,\,x) = e^{-x^2}\,e^{-ix\xi+\frac{\xi^2}{4}}\left[e^{-ix\eta+\frac{\xi\eta}{2}+\frac{\eta^2}{4}} - 1\right].$$

The part outside the square brackets will be fine when we divide by $|\eta|$ and take absolute values, so we look at

$$\frac{e^{-ix\eta+\frac{\xi\eta}{2}+\frac{\eta^2}{4}} - 1}{\eta} = \frac{e^{-ix\eta}e^{\frac{\xi\eta}{2}+\frac{\eta^2}{4}} - 1}{\eta} = \frac{(e^{-ix\eta} - 1 + 1)e^{\frac{\xi\eta}{2}+\frac{\eta^2}{4}} - 1}{\eta}$$

$$= \frac{e^{-ix\eta} - 1}{\eta}\,e^{\frac{\xi\eta}{2}+\frac{\eta^2}{4}} + \frac{e^{\frac{\xi\eta}{2}+\frac{\eta^2}{4}} - 1}{\eta} =: A + B.$$

Then

$$|A| = \left|\frac{e^{-ix\eta} - 1}{\eta}\,e^{\frac{\xi\eta}{2}+\frac{\eta^2}{4}}\right| \le \left|\frac{e^{-ix\eta} - 1}{\eta}\right| e^{\frac{|\xi|}{2}+\frac{1}{4}}$$

$$= \left|e^{-ix\eta/2}\frac{e^{-ix\eta/2} - e^{ix\eta/2}}{\eta}\right| e^{\frac{|\xi|}{2}+\frac{1}{4}} = \left|\frac{e^{-ix\eta/2} - e^{ix\eta/2}}{\eta}\right| e^{\frac{|\xi|}{2}+\frac{1}{4}}$$

$$= 2|\sin(x\eta/2)/\eta| e^{\frac{|\xi|}{2}+\frac{1}{4}} \le |x| e^{\frac{|\xi|}{2}+\frac{1}{4}}.$$

This will be OK, because we'll multiply by a Gaussian. The estimate for B is a little tricky. Let's define $q(\eta) = \frac{\xi\eta}{2} + \frac{\eta^2}{4}$, and notice that $q(0) = 0$. Then we can write

$$\begin{aligned} B &= \frac{e^{\frac{\xi\eta}{2}+\frac{\eta^2}{4}} - 1}{\eta} \\ &= \frac{e^{q(\eta)} - e^{q(0)}}{\eta} \\ &= e^{\zeta}\,\frac{q(\eta) - q(0)}{\eta} \quad \text{(Mean Value Theorem,} \\ &\quad \text{where } \zeta \text{ is between } 0 \text{ and } q(\eta)) \\ &= e^{\zeta}\,\frac{q(\eta)}{\eta} \quad \text{(since } q(0) = 0). \end{aligned}$$

We can find a bound for $|q(\eta)|$, for

$$|q(\eta)| = \left|\frac{\xi\eta}{2} + \frac{\eta^2}{4}\right| \le (|\xi|/2 + 1/4)|\eta|.$$

Thus

$$|B| \le e^{|\zeta|}\left|\frac{q(\eta)}{\eta}\right| \le e^{|q(\eta)|}\,\frac{(|\xi|/2 + 1/4)|\eta|}{|\eta|} \le e^{|\xi|/2+1/4}(|\xi|/2 + 1/4).$$

Finally, then,

$$\left|\frac{F(\xi+\eta,\,x) - F(\xi,\,x)}{\eta}\right| \le e^{-x^2}e^{\xi^2/4}e^{|\xi|/2+1/4}(|x| + |\xi|/2 + 1/4).$$

Since ξ is constant here, the right-hand side function is a polynomial in $|x|$ times a Gaussian, hence it is integrable, as we wished to show.