

We begin with one version of the definition of an equivalence relation.

Definition 1: An equivalence relation on a non-empty set S is a function $R(x, y)$ of two variables x and y , both from the set S , and $R(x, y)$ has the value one if the relation R between x and y is TRUE, and has the value zero if the relation R between x and y is FALSE. In addition, R must satisfy these conditions:

- (a) $R(x, x) = 1$ for all x in S ;
- (b) $R(x, y) = R(y, x)$ for all x and y in S ;
- (c) for all x, y and z in S , if $R(x, y) = 1$ and $R(y, z) = 1$ then $R(x, z) = 1$.

We say that two elements x and y of S are *equivalent modulo R* if $R(x, y) = 1$, and *inequivalent* otherwise.

Examples: The basic example is *equality*. It is always true that $x = x$. This is called *reflexivity*; equality is a *reflexive* relation. It is also true that if $x = y$ then $y = x$. This is called *symmetry*; equality is a *symmetric* relation. Finally, if $x = y$ and $y = z$ then $x = z$. This is called *transitivity*; equality is a *transitive* relation.

Another example: two integers m and n have the same parity if $m - n$ is even. That is, m and n are both even or both odd. As an exercise you can do easily, please check that if we define $R(m, n)$ on $\mathbb{Z} \times \mathbb{Z}$ by: $R(m, n) = 1$ if $m - n$ is even and define $R(m, n) = 0$ if $m - n$ is odd, then R is an equivalence relation.

Yet another example: congruence of triangles.

Our main example: *equal almost everywhere*. We take for S the set of all complex-valued functions defined on the real line \mathbb{R} , and when f and g are in S we define $R(f, g)$ to be one if $f(x) = g(x)$ almost everywhere (a.e.), and define $R(f, g)$ to be zero if the set where $f(x) \neq g(x)$ is not a set of measure zero.

Since $f(x) = f(x)$ everywhere, $R(f, f) = 1$ for all $f \in S$. If $f(x) = g(x)$ a.e., then $g(x) = f(x)$ a.e. because equality is symmetric, so equality almost everywhere is symmetric. Finally, if $f(x) = g(x)$ a.e. and $g(x) = h(x)$ a.e. (where $h \in S$), this means that the set E_1 where $f(x) \neq g(x)$ is a set of measure zero and the set E_2 where $g(x) \neq h(x)$ is a set of measure zero.

Lemma: If E_1 and E_2 are sets of measure zero then $E_1 \cup E_2$ is a set of measure zero.

Proof: Let $\epsilon > 0$ be given. Then there exists a sequence of intervals $I_{1,k}$, $k \geq 1$, and a set of intervals $I_{2,k}$ such that $\sum_k |I_{1k}| < \epsilon/2$ and $\sum_k |I_{2k}| < \epsilon/2$, where $|I|$ denotes the length of an interval I . We can combine these sequences of intervals into one sequence of intervals: $I_{2m-1} := I_{1,m}$ and $I_{2m} := I_{2,m}$. Then $\sum_m |I_m| = \sum_k (|I_{2k-1}| + |I_{2k}|) = \sum_k (|I_{1,k}| + |I_{2,k}|) < \epsilon$.

Having proved the Lemma we can apply it: If there is an $x \in \mathbb{R}$ where $f(x) \neq h(x)$, then either $f(x) \neq g(x)$ or $g(x) \neq h(x)$ (or both). This means that $x \in E_1 \cup E_2$. Thus the set where $f(x) \neq h(x)$ is a set of measure zero, by the Lemma. Hence equality almost everywhere is transitive, and so equality almost everywhere is an equivalence relation.

An equivalence relations divides the set S into subsets of equivalent elements

A subset E of S is an equivalence class

- (1) if $R(x, y) = 1$ for every pair of elements x and y in E and
- (2) if $R(x, y) = 0$ for every $x \in E$ and every $y \notin E$.

The equivalence class containing $x \in S$, denoted \bar{x} , is defined to be the set of all $y \in S$ such that $R(x, y) = 1$.

Theorem: If R is an equivalence relation on the nonempty set S then for all x and y in S , $\bar{x} = \bar{y}$ if and only if $R(x, y) = 1$. In particular, for all x and y in S , exactly one of (1) $\bar{x} = \bar{y}$ and (2) $\bar{x} \cap \bar{y} = \emptyset$ is true. Thus S can be expressed as the disjoint union of equivalence classes, and each equivalence class E is equal to \bar{x} for every $x \in E$. The collection of equivalence classes is denoted S/R .

Proof: If $\bar{x} = \bar{y}$ then $y \in \bar{x}$. Thus $R(x, y) = 1$. If $R(x, y) = 1$, suppose that $z \in \bar{y}$. Then (by definition) $R(y, z) = 1$. By transitivity $R(x, z) = 1$. Thus $z \in \bar{x}$. This is true for all $z \in \bar{y}$ so $\bar{y} \subseteq \bar{x}$. We can do the argument again: if $R(x, y) = 1$, suppose that $w \in \bar{x}$. Then $R(x, w) = 1 = R(w, x)$. By transitivity $R(w, y) = 1 = R(y, w)$. Thus $w \in \bar{y}$ so $\bar{x} \subseteq \bar{y}$ and thus $\bar{x} = \bar{y}$. Suppose that $z \in \bar{x}$ and $z \in \bar{y}$. Then $R(x, z) = 1$ and $R(y, z) = 1 = R(z, y)$ so $R(x, y) = 1$ (why?). Thus if \bar{x} and \bar{y} have a point in common, they coincide. This proves (1) and (2). The rest of the theorem consists of definitions!

An example to illustrate the Theorem

Let's make our set S be \mathbb{Z} , the set of all integers. We choose a positive integer b , and we'll create an equivalence relation based on b . We take it as known that for every integer n there exist unique integers q (quotient) and r (remainder), with $0 \leq r < b$, such that $n = qb + r$. For integers m and n we now define $R(m, n)$ to be one if $m - n$ is divisible by b (has zero remainder when divided by b) and define $R(m, n)$ to be zero if $m - n$ is not divisible by b . Another way to tell that the relation R is TRUE is: m and n have the same remainder when divided by b . For those familiar with some number theory, R is TRUE if m and n are *congruent modulo b* . Then $R(m, n) = 1$ if and only if (notation) $m \equiv n \pmod{b}$.

For each integer m , \bar{m} is the set of all integers of the form $m + kb$, $k \in \mathbb{Z}$. For example, when $b = 3$ there are just 3 equivalence classes: $\{3k : k \in \mathbb{Z}\}$, $\{3k + 1 : k \in \mathbb{Z}\}$, $\{3k + 2 : k \in \mathbb{Z}\}$. In general there are b equivalence classes modulo b . When $b = 2$ the classes are the even integers and the odd integers.

The standard Definition of equivalence relations

We need a few preliminaries. If X and Y are nonempty sets we define their *Cartesian product* to be the set of all ordered pairs (x, y) , where $x \in X$ and $y \in Y$. Notation: $X \times Y$; In terms of "set selector" notation,

$$X \times Y := \{(x, y) : x \in X \text{ and } y \in Y\}.$$

Then a *relation between X and Y* is a subset of $X \times Y$. Very simple, very general, very boring – but nonetheless very useful for making definitions.

A *relation on X* is a relation between X and X , that is, a subset of $X \times X$.

A relation on X is *reflexive* if it contains the *diagonal* of $X \times X$, namely $\{(x, x) : x \in X\}$. Other names for the diagonal: the identity, the relation of equality.

A relation R on X ($R \subseteq X \times X$) is *symmetric* if it contains its reflection, or transpose: $R^T = \{(y, x) : (x, y) \in R\}$. That is, R is symmetric if $R^T = R$.

A relation R on X is *transitive* if $(x, y) \in R$ and $(y, z) \in R$ together imply $(x, z) \in R$.

Definition 2: A relation R on X is called an *equivalence relation* if R is reflexive, symmetric and transitive.

This definition is standard because mathematicians like to (as much as possible) reduce everything to sets (ordered pairs can be defined in terms of sets) and functions. Actually, functions can be defined in terms of sets too.

The equivalence classes in our main example

There is more utility in the equivalence classes modulo equality almost everywhere than is at first obvious: we can add equivalence classes and multiply them too – though this requires definitions.

As we have seen, if f is a complex-valued function defined on \mathbb{R} the equivalence class determined by f is the set of all functions $g : \mathbb{R} \rightarrow \mathbb{C}$ such that $g(x) = f(x)$ almost everywhere. That is, the set of x with $g(x) \neq f(x)$ is a set of measure zero, or a *null set*. But then $f(x) - g(x) = 0$ a.e. and so we can let Z denote the set of all functions $\zeta(x)$ that are zero a.e. This allows us to write the equivalence class determined by f , \bar{f} , as $f + Z$. This is a set of functions, namely all functions of the form $g = f + \zeta$, where $\zeta \in Z$. Our trick is: *regard all the functions in the set $f + Z$ as identical*.

To add two set of functions, say S_1 and S_2 , we imagine writing down all sums $f_1 + f_2$, where $f_1 \in S_1$ and $f_2 \in S_2$. If two such sums are the same function, we keep only one copy of the *sum*, for a set can have only one copy of each of its elements! That is,

$$S_1 + S_2 := \{h : \mathbb{R} \rightarrow \mathbb{C} : \text{there exists } f_1 \in S_1 \text{ and there exists } f_2 \in S_2 \text{ such that } h = f_1 + f_2\}.$$

We recall that if E_1 and E_2 are null sets then $E_1 \cup E_2$ is also a null set. This fact allows us to observe that $Z + Z = Z$. For, if $\zeta_1 \in Z$ and $\zeta_2 \in Z$ then there are null sets E_1 and E_2 such that $\zeta_k(x) = 0$ except in the null set E_k , $k = 1, 2$. Therefore $\zeta_1(x) + \zeta_2(x) = 0$ unless $x \in E_1 \cup E_2$. Note: it *could* be true that $\zeta_1(x) + \zeta_2(x) = 0$

for some (maybe all!) points x in $E_1 \cup E_2$. This still means that $\zeta_1 + \zeta_2 = 0$ a.e. Now we suppose that f and g are complex-valued functions defined on \mathbb{R} . Then by the way we defined addition of sets of functions,

$$\begin{aligned} \overline{f+g} &= (f+Z) + (g+Z) \\ &= \{h : \mathbb{R} \rightarrow \mathbb{C} : \text{there exists } f_1 \in f+Z \text{ and there exists } f_2 \in g+Z \text{ such that } h = f_1 + f_2\} \\ &= \{h : \mathbb{R} \rightarrow \mathbb{C} : \text{there exists } \zeta_1 \in Z \text{ and there exists } \zeta_2 \in Z \text{ such that } h = (f + \zeta_1) + (g + \zeta_2)\} \\ &= \{h : \mathbb{R} \rightarrow \mathbb{C} : \text{there exists } \zeta_1 \in Z \text{ and } \zeta_2 \in Z \text{ such that } h = (f + g) + (\zeta_1 + \zeta_2)\} \\ &= \{h : \mathbb{R} \rightarrow \mathbb{C} : \text{there exists } \zeta \in Z \text{ such that } h = (f + g) + \zeta\} \\ &= (f + g) + Z = \overline{f + g}, \text{ by definition.} \end{aligned}$$

In other words, set addition “respects” addition: the equivalence class of the sum is the sum of the equivalence classes. Note that here the word “sum” is used in two different ways!

If we define the product of two sets of functions the same way, we can also show (tho it takes a little more work) that

$$\overline{f \cdot g} = \overline{f} \cdot \overline{g}.$$

All this lies behind our “everyday” way to work with functions in the “almost-everywhere” context, which is to ignore the fact that a function that is zero a.e. need not be zero at every point, and pretend that it *is* zero at every point, without ever *asserting* that it is zero at every point, and without ever *using* its (zero) value at any particular point.