The Exponential Saga

We are going to prove that for all real x,

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \tag{0.1}$$

exists. It will turn out that the limit function is e^x, the "exponential function" that you brought here from Calculus. But the meaning of the exponential expression e^x is not part of our axioms, nor is the "exponential" expression covered by the "rules" that have been cleared for your use.

The purpose of this note is to use the tools we have developed to prove the properties of the exponential function that you "brought with you."

We will proceed in small steps. In the first two steps we will assume that $x > 0$, and prove that the sequence $\left\{\left(1+\frac{x}{n}\right)^n\right\}_{n=1}^{\infty}$ is, first, strictly increasing, and, second, bounded above. It then follows from Knopp's First Main Test that, with $f_n(x) := \left(1+\frac{x}{n}\right)^n$,

$$f(x) := \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \quad \text{exists if } x>0. \tag{0.2}$$

First Step: $\{f_n(x)\}$ **is strictly increasing**

We will expand $f_n(x) = \left(1+\frac{x}{n}\right)^n$, using the Binomial Theorem, and use inequalities to replace the terms in that Binomial sum with larger terms, finally arriving at the terms in the Binomial expansion of the next function, $f_{n+1}(x)$, except for its last term, another positive term. Thus the *idea* is to replace each term in the Binomial expansion of f_n by the corresponding term in the Binomial expansion for f_{n+1} **(we have to show each new term is larger),** and then add the last, positive, term from f_{n+1}, to show that $f_n(x) < f_{n+1}(x)$. Maybe doing it is easier than describing how it is done:

$$\left(1+\frac{x}{n}\right)^n = \sum_{k=0}^{n}\binom{n}{k}1^{n-k}\left(\frac{x}{n}\right)^k = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{n}\right)^k. \tag{1.1}$$

The quantity we want to arrive at, or nearly at, is

$$\left(1+\frac{x}{n+1}\right)^{n+1} = \sum_{k=0}^{n+1}\binom{n+1}{k}\left(\frac{x}{n+1}\right)^k. \tag{1.2}$$

We will transform each term in (1.1) into the corresponding term of (1.2). The k-th term in $\left(1+\frac{x}{n}\right)^n$ can be transformed (by multiplying by 1 written in a fancy way) into the k-th term in $\left(1+\frac{x}{n+1}\right)^{n+1}$, except for a factor. The steps follow:

$$\begin{aligned}\binom{n}{k}\left(\frac{x}{n}\right)^k &= \frac{n!}{k!(n-k)!}\left(\frac{x}{n}\right)^k = \frac{n!}{k!(n-k)!}\left(\frac{n+1}{n}\right)^k\left(\frac{x}{n+1}\right)^k\\ &= \left(\frac{n+1}{n}\right)^k\frac{n+1-k}{n+1}\,\frac{(n+1)n!}{k!(n+1-k)!}\left(\frac{x}{n+1}\right)^k\\ &= \left(\frac{n+1}{n}\right)^k\frac{n+1-k}{n+1}\,\frac{(n+1)!}{k!(n+1-k)!}\left(\frac{x}{n+1}\right)^k\\ &= \left[\left(\frac{n+1}{n}\right)^k\frac{n+1-k}{n+1}\right]\binom{n+1}{k}\left(\frac{x}{n+1}\right)^k.\end{aligned} \tag{1.3}$$

The "factor" is $\left[\left(\frac{n+1}{n}\right)^k\frac{n+1-k}{n+1}\right]$. The "fancy 1" was $1 = \left(\frac{n+1}{n+1}\right)^k\frac{n+1}{n+1}\frac{n+1-k}{n+1-k}$. What we did was to re-write the k-th term in the sum (1.1) as follows:

$$\binom{n}{k}\left(\frac{x}{n}\right)^k = \left[\left(\frac{n+1}{n}\right)^k\frac{n+1-k}{n+1}\right]\binom{n+1}{k}\left(\frac{x}{n+1}\right)^k. \tag{1.4}$$

In order to show that $\binom{n}{k}\left(\frac{x}{n}\right)^k \le \binom{n+1}{k}\left(\frac{x}{n+1}\right)^k$, we will show that $\left(\frac{n+1}{n}\right)^k \frac{n+1-k}{n+1} \le 1$. We simplify first:

$$\left(\frac{n+1}{n}\right)^k \frac{n+1-k}{n+1} = \left(1+\frac{1}{n}\right)^k \left(1-\frac{k}{n+1}\right).$$

When $k=0$ this is 1. When $k=1$ we have $\left(1+\frac{1}{n}\right)\left(1-\frac{1}{n+1}\right) = \left(\frac{n+1}{n}\right)\left(\frac{n}{n+1}\right) = 1$ again. We can continue now by hoping that each term is smaller than the last one, that is, that

$$\left(1+\frac{1}{n}\right)^{k+1}\left(1-\frac{k+1}{n+1}\right) < \left(1+\frac{1}{n}\right)^{k}\left(1-\frac{k}{n+1}\right) \quad \text{if } 0<k<n. \tag{1.5}$$

To prove that this is true, we will subtract the term on the left in (1.5) from the term on the right:

$$\left(1+\frac{1}{n}\right)^{k}\left(1-\frac{k}{n+1}\right) - \left(1+\frac{1}{n}\right)^{k+1}\left(1-\frac{k+1}{n+1}\right) = \left(1+\frac{1}{n}\right)^{k}\left(1-\frac{k}{n+1} - \left(1+\frac{1}{n}\right)\left(1-\frac{k+1}{n+1}\right)\right).$$

All we have to do is to show that the second factor on the right is positive. This factor simplifies drastically to a simple formula:

$$1-\frac{k}{n+1} - \left(1+\frac{1}{n}\right)\left(1-\frac{k+1}{n+1}\right) = \frac{k}{n(n+1)}.$$

You should do the simplification! We have now proven that for $0 \le k \le n$,

$$\binom{n}{k}\left(\frac{x}{n}\right)^k \le \binom{n+1}{k}\left(\frac{x}{n+1}\right)^k. \tag{1.6}$$

Therefore in (1.1) we can use the inequality (1.6) and nearly reach (1.2):

$$\left(1+\frac{x}{n}\right)^n = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{n}\right)^k \le \sum_{k=0}^{n}\binom{n+1}{k}\left(\frac{x}{n+1}\right)^k. \tag{1.7}$$

We can now add the positive term $\binom{n+1}{n+1}\left(\frac{x}{n+1}\right)^{n+1}$, which is needed to make the sum on the right be equal to $\left(1+\frac{x}{n+1}\right)^{n+1}$, and we obtain

$$f_n(x) = \left(1+\frac{x}{n}\right)^n = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{n}\right)^k < \sum_{k=0}^{n+1}\binom{n+1}{k}\left(\frac{x}{n+1}\right)^k = \left(1+\frac{x}{n+1}\right)^{n+1} = f_{n+1}(x). \tag{1.8}$$

This completes Step 1, proving that our sequence is strictly increasing when $x>0$.

Second Step: $\{f_n(x)\}$ is bounded above

Once again we expand $\left(1+\frac{x}{n}\right)^n$, using the Binomial Theorem, and use inequalities to make the terms in the Binomial sum larger, finally arriving at a new sum, $\sum_{k=0}^{n} x^k/k!$, that we can show is bounded above. We have

$$\left(1+\frac{x}{n}\right)^n = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{n}\right)^k = \sum_{k=0}^{n}\frac{n!x^k}{(n-k)!k!n^k} = \sum_{k=0}^{n}\frac{n!}{(n-k)!n^k}\frac{x^k}{k!}. \tag{2.1}$$

We will follow the same idea we used before, and show that $\dfrac{n!}{(n-k)!n^k} \le 1$. To do so, we write

$$\frac{n!}{(n-k)!n^k} = \frac{n(n-1)\cdots(n-k+1)}{n^k} = \frac{(n-1)\cdots(n-k+1)}{n^{k-1}}.$$

The ratio at the end is less than one if $k > 0$, because it is the product of $k-1$ ratios, each of the form $\frac{n-k}{n} < 1$. If $k = 0$, we have equality instead of strict inequality. Therefore for $n > 0$ and $x > 0$,

$$\left(1 + \frac{x}{n}\right)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!n^k} \frac{x^k}{k!} < \sum_{k=0}^{n} \frac{x^k}{k!}. \tag{2.2}$$

Second Step, continued: bounding $\sum_{k=0}^{n} \frac{x^k}{k!}$ **when** $x > 0$.

We will assume, for convenience, that n is large relative to x. We'll specify later just how large we want n to be. But let's notice now that our final estimate won't depend on our convenient choice of n because if our final bound works for all large n then it works for all n because the sequence $E_n(x) := \sum_{k=0}^{n} \frac{x^k}{k!}$ is easily seen to be strictly increasing, because x is positive. Also, we need to notice that our bound can depend on x. It just cannot depend on n.

We begin with a crucial choice of an integer m, related to x, that will allow us to relate part of the sum that is $E_n(x)$ to a geometric series. Let m be the integer that satisfies the inequalities

$$m - 1 \le x < m, \quad \text{so that} \quad m = [x+1]. \tag{2.3}$$

In words, m is the largest integer that is less than or equal to $x+1$. We will ask that $n > 2m+3$. Then

$$\sum_{k=0}^{n} \frac{x^k}{k!} = \sum_{k=0}^{2m+1} \frac{x^k}{k!} + \sum_{k=2m+2}^{n} \frac{x^k}{k!}. \tag{2.4}$$

On the right, the first sum is a number that depends only on x, not on n. In the second sum we will look at the individual terms now:

$$\frac{x^k}{k!} = \frac{x^{2m+2}}{(2m+2)!} \frac{x^{k-2m-2}}{k(k-1)\cdots(2m+3)}.$$

In passing we note that the second factor is just 1 in case $k = 2m+2$. If $k > 2m+2$ then the second factor has $k - 2m - 2$ factors in the denominator, and each factor is at least $2m+3$. Moreover, if $k > 2m+3$,

$$\frac{x^k}{k!} = \frac{x^{2m+2}}{(2m+2)!} \frac{x^{k-2m-2}}{k(k-1)\cdots(2m+3)} < \frac{x^{2m+2}}{(2m+2)!} \frac{x^{k-2m-2}}{(2m+3)^{k-2m-2}}.$$

We have asked that $n > 2m+3$. Thus for some k in the sum we have strict inequality, and non-strict inequality in all terms. Therefore

$$\sum_{k=0}^{n} \frac{x^k}{k!} < \sum_{k=0}^{2m+1} \frac{x^k}{k!} + \frac{x^{2m+2}}{(2m+2)!} \sum_{k=2m+2}^{n} \frac{x^{k-2m-2}}{(2m+3)^{k-2m-2}} = \sum_{k=0}^{2m+1} \frac{x^k}{k!} + \frac{x^{2m+2}}{(2m+2)!} \sum_{k=2m+2}^{n} \left(\frac{x}{2m+3}\right)^{k-2m-2}. \tag{2.4}$$

There is only one part of the right-hand-side that depends on n, and we want to bound it, by a bound that is independent of n. We can relate this part to a geometric series because $x < m$:

$$\sum_{k=2m+2}^{n} \left(\frac{x}{2m+3}\right)^{k-2m-2} < \sum_{k=2m+2}^{n} \left(\frac{m}{2m+3}\right)^{k-2m-2} < \sum_{k=2m+2}^{n} \left(\frac{1}{2}\right)^{k-2m-2} = \sum_{k=0}^{n-2m-2} \left(\frac{1}{2}\right)^{k}. \tag{2.5}$$

The last step is a "literal" change of variables: k was replaced by $k+2m+2$. We know what the last sum is! Recall the Geometric Series partial-sum formulas, which follow from the Difference-of-Powers Formula: if $u \ne 1$, then $\sum_{k=0}^{k} u^k = \frac{1-u^{k+1}}{1-u}$. In our case, $u = 1/2$ and $k = n - 2m - 2$, so

$$\sum_{k=0}^{n-2m-2} \left(\frac{1}{2}\right)^k = \frac{1-(1/2)^{k+1}}{1-(1/2)} < 2. \tag{2.6}$$

Consequently (2.5) and (2.6) lead to

$$\sum_{k=2m+2}^{n}\left(\frac{x}{2m+3}\right)^{k-2m-2} < \sum_{k=0}^{n-2m-2}\left(\frac{1}{2}\right)^{k} < 2. \tag{2.7}$$

We can now use this in (2.2) and in (2.4) and we get, for all natural numbers n,

$$f_n(x) = \left(1+\frac{x}{n}\right)^n < \sum_{k=0}^{n}\frac{x^k}{k!} < \sum_{k=0}^{2m+1}\frac{x^k}{k!} + 2\frac{x^{2m+2}}{(2m+2)!}. \tag{2.8}$$

This is a bound that depends on x but not on n.

Exercise: Prove that the series $E(x) := \sum_{k=0}^{\infty}\frac{x^k}{k!}$ $\left(= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{x^k}{k!},\ \text{by definition!}\right)$ converges for each $x > 0$.

We have shown that the sequence $\{f_n(x)\}$ converges for every positive x!

We now define a function $f(x)$, defined for $x > 0$, to be the limit:

$$f(x) := \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n.$$

Example: $x = 1$. Here, $m = [x+1] = 2$. Then (2.8) becomes

$$f_n(1) = \left(1+\frac{1}{n}\right)^n < \sum_{k=0}^{n}\frac{1}{k!} < \sum_{k=0}^{5}\frac{1}{k!} + \frac{2}{6!}.$$

We can evaluate the last term on the right:

$$\begin{aligned}
\sum_{k=0}^{5}\frac{1}{k!} + \frac{2}{6!} &= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{360} \\
&= 2 + \frac{60+20+5+1}{120} + \frac{1}{360} \\
&= 2 + \frac{86}{120} + \frac{1}{360} \\
&= 2 + \frac{258}{360} + \frac{1}{360} \\
&= 2 + \frac{259}{360} < 2.7195 \quad \text{(using a calculator).}
\end{aligned}$$

We have in mind that this should converge to the e that we "brought with us;" this is approximately $2.718281\ldots$ Thus our upper bound is pretty good, even though we only needed it to prove convergence.

Since every $f_n(0) = 1$, it is trivial to prove that

the sequence $\{f_n(0)\}$ converges to 1. Thus we define $f(0) := 1$.

The question now arises: what about the convergence when the variable is negative? Let's still keep $x > 0$ and study the sequence $\{f_n(-x)\}$. We will now prove that

the sequence $\{f_n(-x)\}$ converges to $\frac{1}{f(x)}$ for all positive x.

We will use an important fact about limits: if $a_n b_n \to L$ and if $a_n \to M \neq 0$, then $b_n \to L/M$. For n sufficiently large, $a_n \neq 0$. Then we have, for all such sufficiently large n, $b_n = \frac{1}{a_n}\,a_n b_n \to \frac{1}{M}\,L = \frac{L}{M}$.

Because we keep in mind that $f(x)$ should have all the properties of e^x (whatever *that* means), we expect that $f_n(-x)f_n(x)$ will converge to 1. By the Difference-of-Powers Formula,

$$
\begin{aligned}
1 - f_n(-x)f_n(x) &= 1 - \left(1 - \frac{x}{n}\right)^n \left(1 + \frac{x}{n}\right)^n = 1^n - \left(1 - \frac{x^2}{n^2}\right)^n \\
&= \left[1 - \left(1 - \frac{x^2}{n^2}\right)\right] \sum_{k=1}^{n} \left(1 - \frac{x^2}{n^2}\right)^{k-1} \\
&= \frac{x^2}{n^2} \sum_{k=1}^{n} \left(1 - \frac{x^2}{n^2}\right)^{k-1} .
\end{aligned}
\tag{3.1}
$$

We notice first that the right-hand side in (3.1) is positive, and second that each term in the sum in the last line (except the first one) is less than one. Therefore, if $n > x$, then each $\left(1 - \frac{x^2}{n^2}\right)^{k-1} < 1$, so

$$0 < 1 - f_n(-x)f_n(x) = \frac{x^2}{n^2} \sum_{k=1}^{n} \left(1 - \frac{x^2}{n^2}\right)^{k-1} < \frac{x^2}{n^2}\, n = \frac{x^2}{n}.$$

By the Squeeze Theorem, $f_n(-x)f_n(x) \to 1$. We need to show that $f(x) = \lim_{n\to\infty} f_n(x) \neq 0$. We will show more: $f(x) = \lim_{n\to\infty} f_n(x) > f_1(x) = 1 + x > 1$.

We have proven that $f(x) > 1 + x > 1$ if $x > 0$.

Thus by the important fact about limits, $f_n(-x) \to 1/f(x)$, as we asserted.

We are pretty close to calling $f(x)$ an exponential function. One thing we need to do then is to verify that $f(x)$ satisfies some "rules for exponents." Thus if the x in $f(x)$ is to behave as an exponent, it should be true that $f(0) = 1$, which we know, and it should be true that $f(u+v) = f(u)f(v)$ for all real numbers u and v.

Proof that, for all real numbers u and v, $f(u+v) = f(u)f(v)$

We assume that u and v are not both zero; if they were, the truth of the equation would be immediate. As usual we work with the approximations and the Difference-of-Powers Formula:

$$
\begin{aligned}
f_n(u)f_n(v) - f_n(u+v) &= \left(1 + \frac{u}{n}\right)^n \left(1 + \frac{v}{n}\right)^n - \left(1 + \frac{u+v}{n}\right)^n \\
&= \left(1 + \frac{u+v}{n} + \frac{uv}{n^2}\right)^n - \left(1 + \frac{u+v}{n}\right)^n \\
&= \frac{uv}{n^2} \sum_{k=1}^{n} \left(1 + \frac{u+v}{n} + \frac{uv}{n^2}\right)^{n-k} \left(1 + \frac{u+v}{n}\right)^{k-1} .
\end{aligned}
\tag{4.1}
$$

For our convenience, let us assume that n is so large that both of the numbers $1 + \frac{u+v}{n} + \frac{uv}{n^2}$ and $1 + \frac{u+v}{n}$ are positive. We will now use two facts about positive integer powers of positive numbers:

Power Fact: if $0 < s < t$ and k is a positive integer, then $s^k < t^k$.

By the Power Fact, $\left(1 + \frac{u+v}{n} + \frac{uv}{n^2}\right)^{n-k} \leq \left(1 + \frac{|u|+|v|}{n} + \frac{|uv|}{n^2}\right)^{n-k}$ and $\left(1 + \frac{u+v}{n}\right)^{k-1} \leq \left(1 + \frac{|u|+|v|}{n}\right)^{k-1}$.
We can now say that, if n is large enough,

$$|f_n(u)f_n(v) - f_n(u+v)| \leq \frac{|uv|}{n^2} \sum_{k=1}^{n} \left(1 + \frac{|u|+|v|}{n} + \frac{|uv|}{n^2}\right)^{n-k} \left(1 + \frac{|u|+|v|}{n}\right)^{k-1} . \tag{4.2}$$

Once more, by the Power Fact we can increase the terms with exponent $k-1$ so they match the terms with exponent $n-k$, and then combine them:

$$|f_n(u)f_n(v) - f_n(u+v)| \leq \frac{|uv|}{n^2} \sum_{k=1}^{n} \left(1 + \frac{|u|+|v|}{n} + \frac{|uv|}{n^2}\right)^{n-1} . \tag{4.3}$$

The terms in the sum no longer depend on k, so we use that observation, plus the Power Fact, to deduce that

$$|f_n(u)f_n(v) - f_n(u+v)| \le \frac{|uv|}{n^2}\, n \left(1 + \frac{|u|+|v|+|uv|}{n}\right)^{n-1}, \tag{4.4}$$

where we used the truth of $\left(1 + \frac{|u|+|v|}{n} + \frac{|uv|}{n^2}\right) \le \left(1 + \frac{|u|+|v|+|uv|}{n}\right)$ when $n \ge 1$. We need a Two–Powers Fact now.

Two–Powers Fact: *if $s > 1$ and k and ℓ are positive integers such that $k < \ell$, then $s^k < s^\ell$.*

By the Two–Powers Fact, $\left(1 + \frac{|u|+|v|+|uv|}{n}\right)^{n-1} < \left(1 + \frac{|u|+|v|+|uv|}{n}\right)^{n} < f(|u|+|v|+|uv|)$.

We can now simplify (4.4) appropriately:

$$|f_n(u)f_n(v) - f_n(u+v)| < \frac{|uv|}{n} f(|u|+|v|+|uv|). \tag{4.5}$$

By the Squeeze Theorem,

$$0 = \lim_{n\to\infty} |f_n(u)f_n(v) - f_n(u+v)| = |f(u)f(v) - f(u+v)|,$$

so that we have indeed proved that $f(u)f(v) = f(u+v)$ for all real numbers u and v.

Proof that $f(x)$ is strictly increasing, and tends to 0 at $-\infty$ and tends to $+\infty$ at $+\infty$.

If $x < y$ then $f(y) = f((y-x)+x) = f(y-x)f(x) > f(x)$ because $y - x > 0$ and we have shown that $f(x-y) > 1$ if $x - y > 0$. Thus $f(x)$ is strictly increasing. Since $f(-x) = 1/f(x)$, we only need to show that $f(x)$ tends to $+\infty$ at $+\infty$. Let R be a large positive number. Then all we have to do is find some $X > 0$ so large that $f(X) > R$. For then, because f is strictly increasing, $x > X \Rightarrow f(x) > R$ also. This is what we mean by "tends to $+\infty$ at $+\infty$." Our X will be a positive integer N that we choose next.

We make an important observation: $f(n) = f(1)^n$. We also know that $f(1) > f_1(1) = 1 + 1 = 2$. Therefore, $f(n) > 2^n$. Since the set of integer powers of 2 is *not* bounded above, this will show that $f(N) > R$ for some integer N. We take $X := N$.

We verify the observation made above by induction. Certainly $f(1) = f(1)^1$. If it is true that $f(n) = f(1)^n$, then $f(n+1) = f(n)f(1) = f(1)^n f(1) = f(1)^{n+1}$.

On the continuity and differentiability of $f(x)$

Proof of continuity

Suppose y is a real number. To show that $f(x)$ is continuous at y we need to show that, given $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.

We will exploit the rule of exponents.

$$f(x) - f(y) = f(y + (x-y)) - f(y) = f(y)[f(x-y) - 1]. \tag{5.1}$$

We will make sure that $|x - y| < 1$, and we will really prove continuity at zero, using $\epsilon/f(y)$ in place of our given ϵ when we prove the continuity at y. Thus we want to show that for every given $\eta > 0$ there exists $0 < \gamma \le 1$ such that for all real u with $|u| < \gamma$, $|f(u) - f(0)| = |f(u) - 1| < \eta$.

Let us assume that n is so large that $1 + \frac{u}{n} > 0$. Then, with $0 < |u| \le 1$ (if $u = 0$ there is nothing to prove),

$$f_n(u) - 1 = \left(1 + \frac{u}{n}\right)^n - 1 = \frac{u}{n}\sum_{k=1}^{n}\left(1 + \frac{u}{n}\right)^{n-k}$$

so that, by the Power Fact and then by the Two–Powers Fact (used twice),

$$\begin{aligned}|f_n(u)-1| &\le \frac{|u|}{n}\sum_{k=1}^{n}\left(1+\frac{|u|}{n}\right)^{n-k}\\ &< \frac{|u|}{n}\sum_{k=1}^{n}\left(1+\frac{|u|}{n}\right)^{n-1}\\ &= |u|\left(1+\frac{|u|}{n}\right)^{n-1} < |u|\left(1+\frac{|u|}{n}\right)^{n} = |u|f_n(|u|)\\ &< |u|f(|u|) \le |u|f(1), \quad \text{since } 0<|u|\le 1.\end{aligned}$$

Since $f(1)$ is a number (in fact, a number less than 2.8), we can make $|u| < \min\{1,\, \eta/3\} =: \gamma$ and obtain

$$|f_n(u)-1| < |u|f(1) < \eta f(1)/3 < \eta, \quad \text{if } |u|<\gamma.$$

We now let $n \to \infty$, and get

(5.2) $$|f(u)-1| \le |u|f(1) < \eta f(1)/3 < \eta, \quad \text{if } |u|<\gamma.$$

We have shown that $f(x)$ is continuous at $x = 0$. Note: the "min" occured because we need to keep $|u| \le 1$.

To show that $f(x)$ is continuous at y, we apply absolute values in (5.1) to get

(5.3) $$|f(x)-f(y)| = f(y)|f(x-y)-1|.$$

Given $\epsilon > 0$ we let $\eta := \epsilon/f(y)$, choose $\delta = \gamma := \min\{1, \eta/3\} = \min\{1, \epsilon/(3f(y))\}$. We set $u := x - y$. Thus if $|x-y| = |u| < \delta = \gamma$, from (5.2) we can conclude that $|f(x-y)-1| = |f(u)-1| < \eta = \epsilon/f(y)$. When we put this inequality into (5.3) we find that the desired result holds:

$$|x-y|<\delta \Rightarrow |f(x)-f(y)|<\epsilon.$$

Proof of differentiability of $f(x)$, and the differentiation formula $f'(x) = f(x)$

This process of reducing the problem of proving continuity at all points to proving continuity at just one point is useful. We will use it to prove differentiability too. This time, steps will be skipped. Your job is to fill them in.

By definition, the derivative of $f(x)$ exists at x if and only if the *difference quotients* $\dfrac{f(x+h)-f(x)}{h}$ have a limit as $h \to 0$, but with h never being 0. From the expectation that $f'(0) = f(0) = 1$ we proceed as follows:

(6.1) $$\text{First, } \frac{f(x+h)-f(x)}{h} = f(x)\frac{f(h)-1}{h}.$$

We now seek to prove that, as $0 \ne h \to 0$,

$$\frac{f(h)-1}{h} \to 1.$$

If we can do this, we will have shown (by the Product Theorem for limits) that $f'(x)$ exists for all x and that $f'(x) = f(x)$ at each x. We will show that

$$\frac{f(h)-1}{h} - 1 \to 0.$$

As usual we return to the source: the functions f_n, and again we make n as large as we need to (but still finite).

(6.2) $$\begin{aligned}\frac{f_n(h)-1}{h} - 1 &= \frac{\frac{h}{n}\sum_{k=1}^{n}\left(1+\frac{h}{n}\right)^{n-k}}{h} - 1\\ &= \frac{1}{n}\sum_{k=1}^{n}\left(1+\frac{h}{n}\right)^{n-k} - 1\\ &= \frac{1}{n}\sum_{k=1}^{n}\left(1+\frac{h}{n}\right)^{n-k} - \frac{1}{n}\sum_{k=1}^{n}1\\ &= \frac{1}{n}\sum_{k=1}^{n}\left[\left(1+\frac{h}{n}\right)^{n-k} - 1\right].\end{aligned}$$

The Difference-of-Powers Formula was used in the numerator, and a fancy way of writing 1 was used so that we could get the differences in the terms of the sum on the last line. We notice that the sum may as well run from 2 to n, because the term with $k=1$ is zero. We will use the Difference-of-Powers Formula on each of the terms in square brackets. This leads to

$$\frac{f_n(h)-1}{h} - 1 = \frac{h}{n^2}\sum_{k=2}^{n}\left[\sum_{r=0}^{n-k-1}\left(1+\frac{h}{n}\right)^k\right]. \tag{6.3}$$

As we have done before, we work with large n and $0<|h|<1$. We can, as before, use the Power Fact and the Two–Powers Fact to get

$$\begin{aligned}\left|\frac{f_n(h)-1}{h}-1\right| &\le \frac{|h|}{n^2}\sum_{k=2}^{n}\sum_{r=0}^{n-k-1}\left(1+\frac{|h|}{n}\right)^k\\ &< \frac{|h|}{n^2}\left(1+\frac{|h|}{n}\right)^{n-1}\sum_{k=2}^{n}\sum_{r=0}^{n-k-1}1\\ &= \frac{|h|}{n^2}\left(1+\frac{|h|}{n}\right)^{n-1}\sum_{k=2}^{n}(n-k) = \frac{|h|}{n^2}\left(1+\frac{|h|}{n}\right)^{n-1}\sum_{k=2}^{n}k\\ &< \frac{|h|}{n^2}\left(1+\frac{|h|}{n}\right)^{n-1} n(n+1)/2\\ &< |h|\left(1+\frac{|h|}{n}\right)^{n}\frac{(n+1)}{2n} < |h|f(|h|).\end{aligned} \tag{6.4}$$

Now we have proved

$$\left|\frac{f_n(h)-1}{h}-1\right| < |h|f(|h|). \tag{6.5}$$

We let $n\to\infty$. This gives us

$$\left|\frac{f(h)-1}{h}-1\right| \le |h|f(|h|) < |h|f(1). \tag{6.6}$$

Now we can let $h\to 0$ and we get that it is indeed true that $f'(0)=1$.

We can now write e^x in place of $f(x)$. *However*, we do not yet know how to deal with such questions as "Is $(e^x)^r = e^{rx}$?" When we study $\log y$, for $y>0$, we will be able to answer the question!

No mathematical work is ever really finished. There are always new questions that come up. The work we have done here has somewhat justified the idea that our function $f(x)$ " = "e^x, which we intuitively regard as "raising $e := f(1)$ to the power x." But what does this mean? How can we raise e, a number about which we know very little, to a power such as $\sqrt{2}$? How can we raise *other* numbers, such as 2 or 10 to arbitrary real powers? Is every positive number a power of e? All of these are questions that we can answer. I hope the answers will be unsatisfying in the sense that they induce you to delve into them more deeply.