

Prove that if $0 < x < 1$ and $x^2 + y^2 < 1$ and $A > 1$ and

$$(*) \quad |y| \leq \sqrt{1 - A^{-2}} \sqrt{x}(1 - x)$$

$$(**) \quad \text{then, with } z = x + iy, \quad |1 - z| \leq A(1 - |z|).$$

We want

$$(1 - x)^2 + y^2 = |1 - z|^2 \leq A^2(1 - |z|)^2 = A^2(1 + x^2 - 2|z| + y^2) = A^2(1 - |z|)^2 = A^2((1 - x)^2 + 2x - 2|z| + y^2).$$

We do some rearranging; we want

$$2A^2(|z| - x) \leq (A^2 - 1)((1 - x)^2 + y^2).$$

We will actually show that if $(*)$ is true then $2A^2(|z| - x) \leq (A^2 - 1)(1 - x)^2$. The prospect of squaring both sides again daunts. There is another important tool we can use: rationalization. Thus

$$2A^2(|z| - x) = 2A^2(\sqrt{x^2 + y^2} - x) = 2A^2(\sqrt{x^2 + y^2} - \sqrt{x^2}) = \frac{2A^2y^2}{\sqrt{x^2 + y^2} + \sqrt{x^2}} \leq \frac{2A^2y^2}{2x} = \frac{A^2y^2}{x}.$$

We drop the y^2 in the denominator because it can't help us there. Thus we really want to show that

$$\frac{A^2y^2}{x} \leq (A^2 - 1)(1 - x)^2 \quad \text{because if that's true then } (**) \text{ is true.}$$

Once again rearranging we see that what we want is that, if $0 < x < 1$ and $x^2 + y^2 < 1$ and $A > 1$ then

$$y^2 \leq \frac{(A^2 - 1)x(1 - x)^2}{A^2} = (1 - A^{-2})x(1 - x)^2. \quad \text{Hence if } (*) \text{ is true, so is } (**).$$