

Further Problem 1: Due Sept 20

Find a simple necessary and sufficient condition, on sets in a VS, that $A + A = 2A$, and deduce that a convex set C satisfies $C + C = 2C$. Must a set A such that $A + A = 2A$ be convex? What if we add the condition that the intersection $A \cap L$ of any line $L = L(p, q)$ in X be closed in \mathbb{R} ? Note: by “a line L in X ” we mean a set L , determined by distinct vectors p and q , and given by $L(p, q) := \{(1-t)p + tq : t \in \mathbb{R}\}$. Then $A \cap L(p, q)$ determines the set in \mathbb{R} given by $\{t \in \mathbb{R} : (1-t)p + tq \in A\}$, and this is the set that we will always require to be closed in the added condition.

A solution

We have $2A \subseteq A + A$ because $2x = x + x$. Equality thus holds if and only if $\frac{A+A}{2} \subseteq A$.

If x and y belong to a convex set, then $\frac{x+y}{2}$ is also in the set, by definition of “convex set.”

Hence $\frac{C+C}{2} = C$ if C is convex.

The set of rational numbers \mathbb{Q} , as a subset of the real numbers, gives an example of a set that is not convex (because of the existence of irrational numbers between any two rational numbers) and yet satisfies $\mathbb{Q} + \mathbb{Q} = 2\mathbb{Q}$.

Before addressing the last part of the problem, let us study the set

$$\Gamma_A := \{t \in [0, 1] : (1-t)x + ty \in A \text{ whenever } x \in A \text{ and } y \in A\},$$

given that

$$(*) \quad A + A = 2A.$$

We can show that Γ_A contains all the dyadic rationals in $[0, 1]$, namely all numbers of the form $t = m/2^n$, where m and n are non-negative integers and $m \leq 2^n$. This can be shown by induction on n . When $n = 0$, the numbers $m/2^n$ are zero and one, which belong to Γ_A trivially. If all the $m/2^n$, $0 \leq m \leq 2^n$ are in Γ_A , and $t := k/2^{n+1}$, then $t = u/2$ and $u \in \Gamma_A$, so for every $x \in A$ and $y \in A$,

$$(1-t)x + ty = \frac{1}{2}x + \left(\frac{1}{2} - \frac{u}{2}\right)x + \frac{u}{2}y = \frac{1}{2}x + \frac{1}{2}[(1-u)x + uy] = \frac{1}{2}x + \frac{1}{2}y' \in A \text{ by } (*)$$

where $y' = (1-u)x + uy \in A$ since $u \in \Gamma_A$.

Now the last part of the problem is easy because we know that the dyadic rational numbers are dense.

Remark Our demand that $A \cap L(p, q)$ be “closed in \mathbb{R} ” can be refined to take care of the defect that the endpoints of the smallest interval of real numbers that contains $I_A := \{t \in \mathbb{R} : (1-t)p + tq \in A\}$ need not be in I_A even if A is convex. An open ball in $X := \mathbb{R}^n$ is an example. We can simply make explicit exceptions of these points or use a metric on the interval in question that “blows up” at a “missing” endpoint.