Suppose B is a non-empty proper subset of $\mathbb{R}^n$ that is both open and closed. Then B^c is a proper non-empty subset of $\mathbb{R}^n$ that is both closed and open. We will obtain a contradiction by proving that there exists a point of $\mathbb{R}^n$ that is a boundary point of B. Since B and B^c have the same boundary points and both are closed, this will show that $B \cap B^c \neq \emptyset$, which is our desired contradiction.

We select, as we may, points $b \in B$ and $c \in B^c$. We then consider the line segment

$$\overline{bc} = \{(1-t)b + tc : 0 \leq t \leq 1\} \text{ and the set } E := \{t \in [0,\, 1] : p_s := (1-s)b + sc \in B \text{ for all } s \in [0,\, t]\}.$$

The set E is non-empty since $0 \in E$. Since $p_1 = c$, there exists $\delta > 0$ such that $B_\delta(c) \subseteq B^c$. Thus E is bounded above by $1 - \delta$. We let $\tau := \sup E < 1$. There are two cases to consider: (1) $p_\tau \in B$ and (2) $p_\tau \in B^c$. In the first case, for all $\epsilon > 0$ there exists $\sigma > \tau > \sigma - \epsilon$ such that $p_\sigma \in B^c$, so p_τ is a boundary point of B. In the second case, since $p_\tau \in B^c$, we must have $\tau > 0$, so each point $p_{\tau - 1/n}$ (for n large enough) is in B, and so p_τ is a boundary point of B. This completes the proof.