Further Problem 2: Due Oct. 6

Chapter 1, # 13:

We let C denote the VS $C[0,1]$; (C,σ) the metric TVS determined by

$$d(f,g) := \int_0^1 \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}dx;$$

(C,τ) the LCTVS given by the seminorms

$$p_x(f) := |f(x)|, \quad \text{where} \quad x \in [0,1].$$

A solution: each of the 4 parts of the problem will be "quoted" before its solution.

(a) Prove that every τ-bounded set in C is also σ-bounded and that the identity map id: $(C,\tau) \to (C,\sigma)$ therefore carries bounded sets into bounded sets.

This solution is strongly based on real analysis. E is τ-bounded if and only if the function

$$M(x) := \sup_{f\in E} |f(x)| < \infty \quad \text{for all} \quad x \in [0,1].$$

We know that $M(x)$ is lower semi-continuous, hence measurable. Let us write

$$|M > \lambda| := |\{x \in [0,1] : M(x) > \lambda\}|,$$

the Lebesgue measure of the set where M exceeds λ. We will also use the Iverson-Knuth notation, writing $[M(x) > \lambda]$ to denote the characteristic function of $\{x \in [0,1] : M(x) > \lambda\}$.

Given any sequence $\{\lambda_n\} \uparrow \infty$, $[M(x) > \lambda_n]$ decreases pointwise on $[0,1]$ to zero. Since $|M > \lambda_1| \le 1 < \infty$, $|M > \lambda_n| \to 0$ by monotone convergence. Hence $|M > \lambda| \to 0$ as $\lambda \to \infty$.

We need to show that, if E is τ-bounded, then for every $1 > \epsilon > 0$ there exists $t > 0$ such that $E \subseteq tB_\epsilon(0)$, or that

$$f \in E \Rightarrow d(f/t, 0) < \epsilon.$$

Thus, suppose that $f \in E$. Then for each $\lambda > 0$ and $t > 0$,

$$\begin{aligned} d(f/t,0) &= \int_0^1 \frac{|f(x)/t|}{1+|f(x)/t|}dx = \int_0^1 \frac{|f(x)|}{t+|f(x)|}dx \\ &= \int_{M(x)>\lambda} \frac{|f(x)|}{t+|f(x)|}dx + \int_{M(x)\le\lambda} \frac{|f(x)|}{t+|f(x)|}dx \\ &< |M>\lambda| + \int_{M(x)\le\lambda} \frac{|f(x)|}{t+|f(x)|}dx \\ &\le |M>\lambda| + \frac{\lambda}{t+\lambda} \quad \left(\text{since } u \mapsto \frac{u}{1+u} \text{ is an increasing function on } \mathbb{R}^+\right) < \epsilon \end{aligned}$$

if we first choose λ large enough, then choose $t > 2\lambda/\epsilon$. This completes the proof of (a).

(b) Prove that id: $(C,\tau) \to (C,\sigma)$ is nevertheless not continuous, although it is sequentially continuous. Show directly that (C,τ) has no countable base.

To show sequential continuity, suppose $f_n \to f$ in (C,τ). This means (by Ex. 7) that $f_n \to f$ pointwise, so that $d(f_n, f) \to 0$ by Lebesgue's Dominated Convergence Theorem.

To show discontinuity, let us show that $B_{1/2}(0)$ contains no τ-neighborhood of 0. If, on the contrary, there were such a neighborhood V then V would contain a basic neighborhood of the form $V_o := \{f \in C : |f(x_i)| <$

M_i, for $0 \le i \le N_o\}$, where, without loss of generality we may assume that $0 = x_0 < x_1 < \cdots < x_{N_o} = 1$, so that the points x_i form a partition of $[0, 1]$. For each i we single out the "middle two-thirds" of the interval $[x_{i-1}, x_i]$, namely $[x_{i-1}^+, x_i^-]$, where $x_{i-1}^+ = \frac{5}{6}x_{i-1} + \frac{1}{6}x_i$ and $x_i^- = \frac{1}{6}x_{i-1} + \frac{5}{6}x_i$. We can now construct a function $f \in V_o$ that is not in $B_{1/2}(0)$. We require that the graph of f be piecewise linear, that $f(x_i) = 0$ (so that $f \in V_o$), and that $f(x) = 3$ on the middle two-thirds of each interval of our partition. Then

$$d(f, 0) = \int_0^1 \frac{|f(x)|}{1 + |f(x)|} dx > \frac{2}{3}\frac{3}{1+3} = 1/2.$$

We *could* have obtained any ϵ, $0 < \epsilon < 1$, by using a possibly larger portion of each interval in our partition, and by choosing some value other than 3 for f there. Indeed, we shall return to this construction later.

This shows that id: $(C, \tau) \to (C, \sigma)$ is not continuous.

To show directly that (C, τ) has no countable base, we suppose that it does. Then each neighborhood in that base would contain a neighborhood of the form $V_n := \{f \in C : |f(x_{ni})| < M_i, \text{ for } 0 \le i \le N_n\}$, where, without loss of generality we may assume that $0 = x_{n0} < \cdots < x_{nN_n} = 1$, so that the points x_{ni} form partitions of $[0, 1]$. We can even arrange things so that the V_n decrease. Now let us choose finitely many points in $[0, 1]$ that are not among the x_{ni}, and any corresponding neighborhood V of zero. For each n, we can construct a function in V_n that is not in V, by modifying the construction we used before – use "middle intervals" sufficiently long to enclose the points associated with V, and trapezoids tall enough to violate the criteria for membership in V.

(c) Prove that every continuous linear functional on (C, τ) has the form

$$f \mapsto \sum_{i=1}^{n} c_i f(x_i),$$

where the $x_i \in [0, 1]$, the $c_i \in \mathbb{C}$ and n is finite.

The functional given by the formula is bounded by 1 on the neighborhood

$$U := \{f \in C : |f(x_i)| < 1/(n + n|c_i|)\}$$

of zero, so it's continuous.

Now let Λ be a continuous linear functional on (C, τ). We know there exists a neighborhood V of zero on which Λ is bounded in absolute value by 1. Thus there exist N points x_i and N numbers M_i such that the special neighborhood

$$V_o := \{f \in C : |f(x_i)| < M_i, \text{ for } 1 \le i \le N\} \subseteq V.$$

Now suppose that $f \in C$ and $f(x_i) = 0$ for each i. We claim that $\Lambda f = 0$. For, since $\alpha f \in V_o$ for all scalars α, $|\Lambda \alpha f| < 1$ which is impossible if $\Lambda f \ne 0$. Next we define functions $f_i \in C$ that are 1 at x_i but are zero at all $x_j \ne x_i$. Then for an arbitrary $f \in C$, the function

$$g := f - \sum_i f(x_i) f_i \text{ satisfies } g(x_i) = 0, \text{ all } i, \text{ so that } \Lambda g = 0, \text{ or, } \Lambda f = \sum_i (\Lambda f_i) f(x_i),$$

which has the desired form, with $c_i := \Lambda f_i$.

(d) Prove that (C, σ) contains no convex open sets other that $\emptyset$ and C.

This follows the idea used to show that L^p, $0 < p < 1$, has the same property. If V is a non-empty convex open set we may assume that $0 \in V$, so that $B_\epsilon(0) \subseteq V$ for some $\epsilon > 0$. We want to show that $C \subseteq V$. Let $f \in C$. The key idea is that, if $h \in C$ and the support of h has measure $\le \epsilon$, then $d(h, 0) < \epsilon$. This is so since

$$d(h, 0) = \int_0^1 \frac{|h(x)|}{1 + |h(x)|} dx < |\overline{\{x : h(x) \ne 0\}}| \int_0^1 dx \le \epsilon. \text{ Moreover, } d(Nh, 0) < \epsilon, \text{ for all } N \in \mathbb{N}.$$

Let us choose $N > 1/\epsilon$. We can now write

$$f(x) = \sum_{k=0}^{N} \left[\frac{2k-1}{2N} \le x \le \frac{2k+1}{2N} \right] \cos^2(\pi N x) f(x) + \sum_{k=1}^{N} \left[\frac{k-1}{N} \le x \le \frac{k}{N} \right] \sin^2(\pi N x) f(x)$$
$$=: \frac{1}{2N+1} \sum_{k=0}^{N} (2N+1) C_k(x) f(x) + \frac{1}{2N+1} \sum_{k=1}^{N} (2N+1) S_k(x) f(x).$$

Since each $C_k(x)$ and each $S_k(x)$ is continuous and has support with measure $1/N < \epsilon$, each of the functions $(2N+1)C_k(x)f(x)$ and $(2N+1)S_k(x)f(x)$ belongs to $B_\epsilon(0)$, so V, being convex, contains f.

(e) Prove that id: $(C, \sigma) \to (C, \tau)$ is not continuous.

Let V_o be a special basis neighborhood of zero. Then V_o is a *proper* convex open set in (C, τ). If id *were* continuous, then V_o would be open in (C, σ), which contradicts (d).