Further Problem 5: Due Nov 7

Prove or disprove: *If* $E_1 \subseteq \mathbb{R}^n$ *and* $E_2 \subseteq \mathbb{R}^m$ *then* $|E_1 \times E_2|_e = |E_1|_e |E_2|_e$, *where the outer measures shown refer to* $\mathbb{R}^{n+m}$, $\mathbb{R}^n$ *and* $\mathbb{R}^m$ *respectively.*

Solution: It is true that $|E_1 \times E_2|_e = |E_1|_e |E_2|_e$. This solution owes much to people in the class, and to Hongjie Dong, who suggested key ideas in the two paragraphs following the next one.

Let us begin with bounded sets E_1 and E_2, that, without loss of generality, we may suppose have positive outer measure. Given $0 < \epsilon < 1$, there exists a bounded open set $G \supseteq E_1 \times E_2$ such that $|G| < |E_1 \times E_2|_e + \epsilon$.

We will find an open set $\mathcal{O} \subseteq G$ with $|\mathcal{O}| > (1-\epsilon)|E_1|_e|E_2|_e$. By letting $\epsilon \to 0$ we then obtain the estimate $|E_1 \times E_2|_e \geq |E_1|_e|E_2|_e$. As we have already shown that $|E_1 \times E_2|_e \leq |E_1|_e|E_2|_e$, and since we know that equality holds (using the convention $0 \cdot \infty = 0$) if at least one of $|E_1|_e$ and $|E_2|_e$ is zero, the proof will be complete.

For each $x \in E_1$ the set $\tau_x(G) \subseteq \mathbb{R}^m$ defined by $\tau_x(G) := \{y \in \mathbb{R}^m : (x, y) \in G\}$ is open and contains E_2. Thus there exists a compact set $K_x \subseteq \tau_x(G)$ such that $|K_x| > (1-\epsilon)|E_2|_e$. Let us use the maximum norm on $\mathbb{R}^{n+m}$ instead of the Euclidean norm. Then the function $g(y) := \operatorname{dist}((x, y), G^c)$, defined using the maximum norm, is continuous on $\mathbb{R}^m$ and has a positive minimum $\delta(x)$ on K_x. Thus for all $y \in K_x$ the open cube with center (x, y) and edge $2\delta(x)$ ("radius" $\delta(x)$ as a ball with respect to the maximum norm) is contained in G.

For convenience we will use some "generic" notation: in *any* space $\mathbb{R}^k$ we will let U denote the open cube with center zero and edges of length two parallel to the coordinate axes. Thus, the "unit" cube $U \subseteq \mathbb{R}^{n+m}$ can be written $U = U \times U$, where the first $U \subseteq \mathbb{R}^n$ and the second one is contained in $\mathbb{R}^m$. We do the same with $Q := \overline{U}$.

Then for every $(x, y) \in \{x\} \times K_x$, $G \supseteq (x, y) + \delta(x)U = (x + \delta(x)U) \times (y + \delta(x)U)$, so $(x + \delta(x)U) \times G(x) \subseteq G$, where

$$G(x) := \bigcup_{y \in K_x} y + \delta(x)U = K_x + \delta(x)U \quad \text{is open in } \mathbb{R}^m.$$

It would be nice if we could find a countable disjoint family of the sets $x + \delta(x)U$ that covers E_1, because then the sets $(x + \delta(x)U) \times G(x)$ from that family would be disjoint and measurable, have union contained in G, and we could use the truth of the product formula for measurable sets to reach the desired conclusion. This is, unfortunately, too much to hope for. However, there is a good substitute if we modify our collection $\{x + \delta(x)U\}$ so that it covers E_1 in the sense of Vitali.

The family of cubes $\{x + \frac{\delta(x)}{k}Q : x \in E_1 \text{ and } k \in \mathbb{Z}^+\}$ covers E_1 in the Vitali sense, so there exist countably many points $x_j \in E_1$ and positive integers k_j such that the cubes $x_j + \frac{\delta(x_j)}{k_j}Q$ are disjoint and $E_1 \setminus \bigcup_j x_j + \dfrac{\delta(x_j)}{k_j}Q$ is a null set. It follows that $|E_1|_e \leq \sum_j |x_j + \frac{\delta(x_j)}{k_j}U|$. The sets $(x_j + \frac{\delta(x_j)}{k_j}U) \times G(x_j)$ are disjoint as well. Since they are each contained in G we have

$$|E_1 \times E_2|_e + \epsilon > |G| \geq \left| \bigcup_j \left(x_j + \frac{\delta(x_j)}{k_j}U \right) \times G(x_j) \right| =: |\mathcal{O}| \geq \sum_j \left| x_j + \frac{\delta(x_j)}{k_j}U \right| (1-\epsilon)|E_2|_e \geq (1-\epsilon)|E_1|_e|E_2|_e.$$

Thus the equality holds if E_1 and E_2 are both bounded and both have positive outer measure. To handle the unbounded case we apply the bounded case to $E_1 \cap B_R(0)$ and $E_2 \cap B_R(0)$ (for R large enough) and let $R \to \infty$, applying the Monotone Convergence Theorem for the outer measures of increasing sequences of sets.

If $|E_1 \times E_2|_e = 0$ it is therefore true that at least one of the sets is a null set. Otherwise we have a contradiction.

Remark: If $H_1 \supseteq E_1$ and $H_2 \supseteq E_2$ are G_δ's with the same outer measures, respectively, that E_1 and E_2 have, then $H_1 \times H_2$ is a G_δ containing $E_1 \times E_2$ and now $|H_1 \times H_2| = |E_1|_e|E_2|_e = |E_1 \times E_2|_e$. But is there always a pair of G_δ's $\widetilde{H}_1 \supseteq E_1$ and $\widetilde{H}_2 \supseteq E_2$ such that $\widetilde{H}_1 \times \widetilde{H}_2 \subseteq G$? If so the proof would be shorter and (possibly) avoid the use of Vitali's Lemma, with elementary proof, in contrast to the proof in Halmos' *Measure Theory* (found by J. Dorfmeister), that uses Fubini's Theorem, a more "advanced" tool.