This solution is not as elegant as some turned in. It is, however, straightforward. Let V be a finite-dimensional subspace of a normed space X, with basis $\mathcal{B} = \{u_1, \ldots, u_m\}$. We may suppose that $\|u_k\| = 1$ for each k. Given $v_n \to x$, with each $v_n \in V$, we need to show that $x \in V$.

Since $v_n =: \sum_{k=1}^m c_{nk}\, u_k$, it would be enough to know that each of the m sequences $\{c_{nk}\}$ converges, for then $\|v_n - \sum c_k u_k\| \le \sum |c_{nk} - c_k| \to 0$, where $c_k := \lim_n c_{nk}$. However, we only know that $\|\sum a_k u_k\| \le \sum |a_k|$, and we would need the inequality in the *other* direction.

If $v = \sum c_k u_k$, let $C(v) := (c_1, \ldots, c_m) \in \mathbb{C}^m$. We want to show that $\max_k |c_k| \le M\|v\|$, for this would allow us to show that each of the $\{c_{nk}\}$ is Cauchy.

Suppose not. Then there exist v_n with $\|v_n\| = 1$ such that $\gamma_n := \max_k |c_{nk}| > n$. We now define $w_n := v_n/\gamma_n$, so that $\|w_n\| < 1/n$. For each n there exists k_n such that $|c_{nk_n}| = \gamma_n$ so there exists k_o such that $|c_{nk_o}| = \gamma_n$ for infinitely many n. Among these n there is a subsequence that converges to a number ω_{k_o} with $|\omega_{k_o}| = 1$ since the unit circle in the complex plane is compact. For all other k and all n, $|\omega_{nk}| = |c_{nk}|/\gamma_n \le 1$. Thus there exist further subsequences that converge; we find n_ℓ such that

$$C(w_{n_\ell}) = C(v_{n_\ell}/\gamma_{n_\ell}) \to (\omega_1, \ldots, \omega_m) \quad \text{and} \quad \omega_{k_o} \ne 0.$$

Hence, by the inequality we *do* know, $w_{n_\ell} \to w := \sum \omega_k u_k \ne 0$. This contradicts what we already knew: $w_n \to 0$. Thus the map $C : V \to \mathbb{C}^m$ is continuous, and our original idea works.