Math 8602, Spring 2004 The Urysohn's Lemma open sets V_r

Let $\mathcal{P}$ denote the collection of all partitions $\pi\,|[0,\,1]$ such that $\pi \subseteq \mathbb{Q}$, let $r : \mathbb{N} \to \mathbb{Q}_0 := \mathbb{Q} \cap [0,\,1]$ be an enumeration with $r_0 = r(0) = 0$ and $r_1 = r(1) = 1$. We let π_n denote the partition that is determined by $\{r_i : 0 \le i \le n+1\}$, and we will denote the *points* of π_n by x_j or x_{nj} or even $x(j)$, $0 \le j \le n+1$.

We let $\mathcal{T}$ denote the topology of X, and define $\mathcal{C} := \{V \in \mathcal{T} : \overline{V}$ is compact$\}$. We will use this notation: $A \subset\subset B$ means that $\overline{A} \subseteq B$. We will work with a well-ordering of $\mathcal{C}$.

Let $\mathcal{V}$ denote the collection of all possible mappings $v : \pi \to \mathcal{C}$ (where $\pi \in \mathcal{P}$) that satisfy this condition:

$$v(1) \subset\subset v(0) \text{ and } 0 < i < n_\pi \Rightarrow v(1) \subset\subset v(x_i) \subset\subset v(x_{i-1}) \subset\subset v(0) \text{ (if } i = 1 \text{ omit the } i-1 \text{ term).}$$

Each $v \in \mathcal{V}$ is actually a *pair*, the partition and the map, so we say $\pi = \pi(v)$. We define a mapping $p : \mathcal{V} \times \mathbb{Q}_0 \to \mathcal{V}$ as follows:

$$p(v,r) := \begin{cases} v & \text{if } r \in \pi(v), \\ \tilde{v}, & \text{if } r \notin \pi(v), \end{cases}$$

where $\tilde{v} : \pi \cup \{r\} \to \mathcal{C}$ is defined by $\tilde{v}(x) = v(x)$ if $x \in \pi(v)$ and $\tilde{v}(r)$ is the first element V of $\mathcal{C}$ that satisfies $v(x_i) \subset\subset V \subset\subset v(x_{i-1})$, where $x_{i-1} < r < x_i$. Then we define $\pi(\tilde{v}) := \pi \cup \{r\}$. We have already seen ($V_0$ and V_1) that $\mathcal{V}$ is nonempty, but there are $\pi(v)$ with arbitrarily large (finite) numbers of elements.

To see this, given an arbitrary $\pi \in \mathcal{P}$ the map $p(v,\,r)$ can be used $n_\pi - 1$ times to construct a unique element $v \in \mathcal{V}$ with $\pi(v) = \pi$: we start with the sets V_0 and V_1 we constructed to begin the proof of Urysohn's Lemma: $K \subset\subset V_1 \subset\subset V_0 \subset\subset U$, where we selected V_0 and V_1 as the *first* elements of $\mathcal{C}$ that (respectively) satisfied $K \subset\subset V_0 \subset\subset U$, then $K \subset\subset V_1 \subset\subset V_0$. Our map v_0 is thus $v_0 : \{0,\,1\} \to \mathcal{C}$ by $v_0(0) := V_0$ and $v_0(1) := V_1$. Then we use $v_1 := p(v_0, x_1)$, and so on, to add the "interior" members of π *in increasing order*. There may, of course, be (many) other $v \in \mathcal{V}$ with $\pi(v) = \pi$. Indeed, the procedure Rudin uses is not what we just did. It was done, though, to show that $\mathcal{V}$ is a large collection! We need another step to construct Rudin's sequence V_{r_i}.

We next define a function $\rho : \mathcal{V} \to \mathcal{V}$ that uses the enumeration r as a "parameter." Given $v \in \mathcal{V}$ we let $i(v)$ be the first i such that $r_i \notin \pi(v)$ and set $\rho(v) := p(v,\, r(i(v))\,)$. By the Recursion Theorem there is a unique sequence $\{v_n\}$ in $\mathcal{V}$ such that (the graph of) v_0 is $\{(0,\,V_0),\,(1,\,V_1)\}$ and for $n \ge 0$, $v_{n+1} = \rho(v_n)$.

A routine induction shows that $\pi(v_n) = \pi_n$. The first m such that $r_n \in \pi_m = \pi(v_m)$ is $n-1$ and there is a unique k_n such that $r_n = x(k_n) \in \pi_{n-1}$. We define $V_{r_n} := v_{n-1}(x(k_n))$.

Finally, let $0 < r < s < 1$. There is a first m such that $\{r,\,s\} \subseteq \pi_m$. Since $\subset\subset$ is transitive it follows that $\overline{V_s} \subseteq V_r$.