Math 5467 Spring '00 The Fourier Transform exists in L^2

The Fourier transform of an integrable function $f(x)$ is

$$\hat{f}(\xi) = \int f(x) e^{-i\xi x} dx, \tag{1}$$

and (by putting bars under the integral sign)

$$\left|\hat{f}(\xi)\right| \le \int |f(x)|\, dx = \|f\|_1.$$

We often say "$f \in L^1$" when f is integrable, and we often write $\|f\|_1$ for $\int |f(x)|\, dx$.

When $f(x)$ is in L^2, the integral may not exist (as a Lebesgue integral).
Example: $f(x) = \dfrac{1}{1+ix}$ is in L^2 but the integral $\displaystyle\int \frac{e^{-i\xi x}}{1+ix} dx$ is not a valid Lebesgue integral, because the absolute value of the integrand, $\dfrac{1}{\sqrt{1+x^2}}$, is not integrable.

Nevertheless, an operation on L^2 functions does exist that plays the rôle of Fourier transform and *that* transform agrees with the L^1 version in case $f(x)$ is in both of L^1 and L^2!

What we will do is to multiply an L^2 function $f(x)$ by $e^{-\epsilon x^2}$, where $\epsilon > 0$, thereby creating a function that is in both of L^1 and L^2. Thus we set

$$f_\epsilon(x) := f(x) e^{-\epsilon x^2}$$

We can also think of f_ϵ as the value of a function defined on the positive real axis, whose value at $\epsilon > 0$ is $f_\epsilon \in L^2(\mathbb{R})$. We can use the Monotone Convergence Theorem, one of the (Useful) Lebesgue Facts, to show that

$$L^2\text{-}\lim_{\epsilon \to 0} f_\epsilon = f, \quad \text{and that} \quad L^2\text{-}\lim_{\epsilon \to \infty} f_\epsilon = 0.$$

We will show that when we take the Fourier transform of f_ϵ, which is

$$\widehat{f_\epsilon}(\xi) = \int e^{-\epsilon x^2} f(x) e^{-i\xi x} dx, \tag{2}$$

and let $\epsilon \to 0$, the limit of the functions $\widehat{f_\epsilon}$ exists in L^2. We will then temporarily use the notation $\mathcal{F}f$ to stand for the limit:

$$\mathcal{F}f := L^2\text{-}\lim_{\epsilon \to 0} \widehat{f_\epsilon}.$$

We will soon thereafter be able to convince ourselves that we may as well use the notation $\hat{f}$ for $\mathcal{F}f$, and view its definition by an integral as a convenient one when possible, but not necessary in applications, because we can always work with $\widehat{f_\epsilon}$ and then take a limit.

The way we will show that the limit exists is to use the completeness of $L^2(\mathbb{R})$. Completeness says that Cauchy sequences converge. That is, if $\|f_n - f_m\| \to 0$ as the minimum of m and n tends to infinity, then there exists a function $f \in L^2$ such that $\|f_n - f\| \to 0$ as n tends to infinity. The limit, f, is unique. We convert the convergence of sequences to the convergence of L^2-valued functions of a real variable by using the "sequences are good enough" principle. This says that in any situation where limits are defined using distances (in our case here, our functions are L^2-valued functions of a real variable, so limits such as $t \to 0$ and $f_t \to f$, both of which are measured in terms of norms (absolute value is a norm!), which are distances, *are* defined using distances), "continuous" limits exist if and only if "sequential" limits exist. So we *could* work as follows: show that, for *every* sequence $0 < \epsilon_n \to 0$,

$$\|\widehat{f_{\epsilon_n}} - \widehat{f_{\epsilon_m}}\| \to 0 \quad \text{as the minimum of } n \text{ and } m \to \infty.$$

Then, for each sequence, a limit would exist that maybe depended on the sequence used. But the *"every* sequence" condition and the "sequences are good enough" principle guarantee that the limit does not depend on the sequence used, so it is safe to conclude that (when true)

$$\text{“}\|\widehat{f_\epsilon} - \widehat{f_\eta}\| \to 0 \quad \text{as the minimum of } \epsilon \text{ and } \eta \to 0\text{”}$$

causes the existence of the limit

$$\mathcal{F}f := L^2\text{-}\lim_{\epsilon\to 0} \widehat{f}_\epsilon.$$

Therefore, we will show that

$$\text{“}\|\widehat{f}_\epsilon - \widehat{f}_\eta\| \to 0 \text{ as the minimum of } \epsilon \text{ and } \eta \to 0\text{”}$$

is true, and that will show (by Cauchy-sequence stuff) that $\mathcal{F}f$ exists and that

$$\|\widehat{f}_\epsilon - \mathcal{F}f\| \to 0 \text{ as } \epsilon \to 0.$$

We will also show (it will be a result of our work with the Fourier transform of a Gaussian) that $\|\mathcal{F}f\|^2 = 2\pi\|f\|^2$. Even more is true! We actually have $\langle \mathcal{F}f, \mathcal{F}g\rangle = 2\pi\langle f, g\rangle$ for all f and g in L^2.

Let's get to work. From (2) and Fubini's Theorem (a Useful Fact),

$$\begin{aligned}\left|\widehat{f}_\epsilon(\xi)\right|^2 &= \int e^{-\epsilon x^2} f(x) e^{-i\xi x}dx \int e^{-\epsilon y^2}\overline{f(y)}e^{i\xi y}dy \\ &= \int\int e^{-\epsilon(x^2+y^2)} f(x)\overline{f(y)}e^{-i\xi(x-y)}dx\,dy.\end{aligned} \tag{3}$$

We will show that $\left|\widehat{f}_\epsilon(\xi)\right|^2$ is integrable in a somewhat sneaky way: we'll multiply by another Gaussian and take a limit, all the while keeping ϵ fixed. We already know that $\left|\widehat{f}_\epsilon(\xi)\right|^2$ is bounded, so for $\gamma > 0$, the function $e^{-\gamma\xi^2}\left|\widehat{f}_\epsilon(\xi)\right|^2$ is an integrable function of ξ. We therefore can work as follows:

$$\begin{aligned}&\int e^{-\gamma\xi^2}\left|\widehat{f}_\epsilon(\xi)\right|^2 d\xi = \\ &= \int e^{-\gamma\xi^2}\left(\int\int e^{-\epsilon(x^2+y^2)} f(x)\overline{f(y)}e^{-i\xi(x-y)}dx\,dy\right)d\xi \\ &= \int\int e^{-\epsilon(x^2+y^2)} f(x)\overline{f(y)}\left(\int e^{-\gamma\xi^2}e^{-i\xi(x-y)}d\xi\right)dx\,dy\end{aligned} \tag{4}$$

by Fubini's Theorem. The innermost integral is the Fourier transform of $e^{-(\sqrt{\gamma}\xi)^2}$, evaluated at $(x-y)$, which we recall is equal to

$$\frac{\sqrt{\pi}}{\sqrt{\gamma}}\, e^{-(x-y)^2/4\gamma}.$$

Now this somewhat resembles the approximate identity based on the Gaussian that we studied. Let's tinker with it to force part of it to be an approximate identity. We have

$$\frac{\sqrt{\pi}}{\sqrt{\gamma}}\, e^{-(x-y)^2/4\gamma} = 2\pi\frac{e^{-(x-y)^2/4\gamma}}{\sqrt{4\pi\gamma}}\,.$$

We can now rewrite (4) as

$$\begin{aligned}&\int e^{-\gamma\xi^2}\left|\widehat{f}_\epsilon(\xi)\right|^2 d\xi = \\ &= 2\pi\int\int e^{-\epsilon(x^2+y^2)} f(x)\overline{f(y)}\left(\frac{e^{-(x-y)^2/4\gamma}}{\sqrt{4\pi\gamma}}\right)dx\,dy \\ &= 2\pi\int e^{-\epsilon x^2} f(x)\overline{\left(\int e^{-\epsilon y^2} f(y)\frac{e^{-(x-y)^2/4\gamma}}{\sqrt{4\pi\gamma}}\,dy\right)}dx.\end{aligned} \tag{5}$$

As $\gamma \to 0$, the inner integral under the bar converges in L^2 to the function $e^{-\epsilon y^2} f(y)$, evaluated at x, so we get, remembering the bar, that

$$\begin{aligned}\lim_{\gamma\to 0}\int e^{-\gamma\xi^2}\left|\widehat{f}_\epsilon(\xi)\right|^2 d\xi &= 2\pi\int e^{-2\epsilon x^2} f(x)\overline{f(x)}dx \\ &= 2\pi\int e^{-2\epsilon x^2}|f(x)|^2 dx.\end{aligned} \tag{6}$$

By the Monotone Convergence Theorem (another Useful Fact),

$$\lim_{\gamma\to 0}\int e^{-\gamma\xi^2}\left|\widehat{f}_\epsilon(\xi)\right|^2 d\xi = \int\left|\widehat{f}_\epsilon(\xi)\right|^2 d\xi.$$

Therefore $\widehat{f}_\epsilon \in L^2$ and

$$\int\left|\widehat{f}_\epsilon(\xi)\right|^2 d\xi = 2\pi\int e^{-2\epsilon x^2}|f(x)|^2 dx. \tag{7}$$

We can at least say now that

$$\lim_{\epsilon\to 0}\int\left|\widehat{f}_\epsilon(\xi)\right|^2 d\xi = 2\pi\lim_{\epsilon\to 0}\int e^{-2\epsilon x^2}|f(x)|^2 dx = 2\pi\int|f(x)|^2 dx. \tag{8}$$

We would like to put the "$\lim\limits_{\epsilon\to 0}$" inside the absolute values in the term on the left-hand side of (7), but we cannot quite do that yet – we need to have a function in L^2 that $\widehat{f}_\epsilon(\xi)$ converges to as $\epsilon \to 0$. Next we will work toward using completeness, in order to show that there exists a function in L^2 that $\widehat{f}_\epsilon(\xi)$ converges to as $\epsilon \to 0$. We will be able to re-use the argument we just did!

Our objective now is to show that

$$\|\widehat{f}_\epsilon - \widehat{f}_\eta\| \to 0 \text{ as the minimum of } \epsilon \text{ and } \eta \to 0.$$

As usual, we will work with the square of the norm. As in the last argument (see (3)),

$$\begin{aligned}&\left|\widehat{f}_\epsilon(\xi) - \widehat{f}_\eta(\xi)\right|^2 \\ &\quad= \int\left(e^{-\epsilon x^2} - e^{-\eta x^2}\right)f(x)e^{-i\xi x}dx\int\left(e^{-\epsilon y^2} - e^{-\eta y^2}\right)\overline{f(y)}e^{i\xi y}dy \\ &\quad= \int\int\left(e^{-\epsilon x^2} - e^{-\eta x^2}\right)\left(e^{-\epsilon y^2} - e^{-\eta y^2}\right)f(x)\overline{f(y)}e^{-i\xi(x-y)}dx\,dy.\end{aligned} \tag{9}$$

We can repeat the steps that led from (3) to (7), replacing each occurrence of $e^{-\epsilon x^2}$ with $\left(e^{-\epsilon x^2} - e^{-\eta x^2}\right)$, and similarly for $e^{-\epsilon y^2}$. The result is the new version of (7):

$$\int\left|\widehat{f}_\epsilon(\xi) - \widehat{f}_\eta(\xi)\right|^2 d\xi = 2\pi\int\left(e^{-\epsilon x^2} - e^{-\eta x^2}\right)^2|f(x)|^2 dx. \tag{10}$$

The integral on the right tends to zero as the minimum of ϵ and $\eta \to 0$. Let us verify this statement. Without loss of generality, we may assume that $\eta < \epsilon$. Then,

$$\left(e^{-\epsilon x^2} - e^{-\eta x^2}\right)^2 = \left(e^{-\eta x^2} - e^{-\epsilon x^2}\right)^2 \le \left(1 - e^{-\epsilon x^2}\right)^2,$$

so

$$\int\left|\widehat{f}_\epsilon(\xi) - \widehat{f}_\eta(\xi)\right|^2 d\xi \le 2\pi\int\left(1 - e^{-\epsilon x^2}\right)^2|f(x)|^2 dx.$$

For each fixed $x \neq 0$, $1 - e^{-\epsilon x^2}$ decreases monotonically to zero as ϵ decreases monotonically to zero. Therefore, the Monotone Convergence Theorem assures us that the integral on the right converges to zero as ϵ decreases monotonically to zero. In other words, given any $\delta > 0$, there exists $\epsilon_o > 0$ such that for all ϵ and η with $\epsilon < \epsilon_o$,

$$\int \left|\widehat{f}_\epsilon(\xi) - \widehat{f}_\eta(\xi)\right|^2 d\xi \leq 2\pi \int \left(1 - e^{-\epsilon x^2}\right)^2 |f(x)|^2 dx < \delta. \tag{11}$$

It follows that

$$\|\widehat{f}_\epsilon - \widehat{f}_\eta\| \to 0 \text{ as the minimum of } \epsilon \text{ and } \eta \to 0.$$

Therefore

$$\mathcal{F}f = L^2\text{-}\lim_{\epsilon \to 0} \widehat{f}_\epsilon$$

exists, and by (8) we have the **Parseval Formula,**

$$\int \left|\mathcal{F}f(\xi)\right|^2 d\xi = 2\pi \int |f(x)|^2 dx. \tag{12}$$

Why the limit may as well be called $\hat{f}$

First, let's look at what happens if $f \in L^1 \cap L^2$. In that case,

$$\widehat{f}_\epsilon(\xi) = \int e^{-\epsilon x^2} f(x) e^{-i\xi x} dx,$$

which (by Dominated Convergence and the "sequences are good enough" principle) converges *pointwise* to $\hat{f}(\xi)$.

At the same time, $\widehat{f}_\epsilon \to \mathcal{F}f$ in L^2. So we ask, does this mean (as we would expect) $\hat{f} = \mathcal{F}f$ if $f \in L^1 \cap L^2$? The answer is *Yes!* and we can verify it by using Fatou's lemma, with ϵ in place of n. In Fatou's Lemma, we take $g(x) = 0$, and we take for $f_\epsilon(x)$ the functions $\left|\mathcal{F}f - \widehat{f}_\epsilon\right|^2$. Then $f_\epsilon(x) \geq g(x)$ and $g(x)$ is integrable, so

$$\int_E \liminf_{\epsilon \to 0} f_\epsilon(x)\, dx \leq \liminf_{\epsilon \to 0} \int_E f_\epsilon(x)\, dx.$$

In other "words,"

$$\int_E \liminf_{\epsilon \to 0} \left|\mathcal{F}f(x) - \widehat{f}_\epsilon(x)\right|^2 dx \leq \liminf_{\epsilon \to 0} \int_E \left|\mathcal{F}f(x) - \widehat{f}_\epsilon(x)\right|^2 dx.$$

On the right we have zero, so a.e.

$$0 = \liminf_{\epsilon \to 0} \left|\mathcal{F}f(x) - \widehat{f}_\epsilon(x)\right|^2 = \left|\mathcal{F}f(x) - \hat{f}(x)\right|^2.$$

Therefore, when we can define it two ways, the Fourier transform of a function in $L^1 \cap L^2$ is the same function (at least, a.e.), defined either way.

The linearity of the Fourier transform; the Plancherel formula

We now write

$$\widehat{f} = L^2\text{-}\lim_{\epsilon \to 0} \widehat{f}_\epsilon, \text{ so that}$$

$$\widehat{cf} = L^2\text{-}\lim_{\epsilon \to 0} \widehat{(cf)}_\epsilon, = cL^2\text{-}\lim_{\epsilon \to 0} \widehat{f}_\epsilon = c\widehat{f} \text{ and } \widehat{f+g} = L^2\text{-}\lim_{\epsilon \to 0} (\widehat{f+g})_\epsilon, = L^2\text{-}\lim_{\epsilon \to 0} \widehat{f}_\epsilon + L^2\text{-}\lim_{\epsilon \to 0} \widehat{g}_\epsilon = \widehat{f} + \widehat{g}.$$

In (12) let us replace $\mathcal{F}f$ by $\widehat{(f + \omega g)}$ on the left, so we must replace f on the right by $f + \omega g$. Next, let us expand both sides, using the complex-number formula $|z + w|^2 = |z|^2 + 2\mathrm{Re}\, z\overline{w} + |w|^2$. We get

$$\int |\widehat{f}(\xi)|^2 d\xi + \int \mathrm{Re}\, \overline{\omega} \widehat{f}(\xi) \overline{\widehat{g}(\xi)}\, d\xi + |\omega|^2 \int |\widehat{g}(\xi)|^2 d\xi = 2\pi \int |f|^2 dx + 2\pi \int \mathrm{Re}\, \overline{\omega} f \overline{g}\, dx + 2\pi |\omega|^2 \int |g|^2 dx.$$

By using (12) again we can cancel the first and last terms on both sides of this equation and so we have

$$\int \mathrm{Re}\,\overline{\omega}\widehat{f}(\xi)\overline{\widehat{g}(\xi)}\,d\xi = 2\pi \int \mathrm{Re}\,\overline{\omega} f(x)\overline{g(x)}\,dx.$$

Let us do two steps now, replacing ω first by 1 and then by i, use the complex-number formula $\mathrm{Re}\,(-i)z = \mathrm{Im}\,z$, and add the results, to obtain the **Plancherel formula:**

$$\int \widehat{f}(\xi)\overline{\widehat{g}(\xi)}\,d\xi = 2\pi \int f(x)\overline{g(x)}\,dx. \tag{13}$$

Finally, we can write the Plancherel formula as an equation involving inner products:

$$\left\langle \widehat{f},\, \widehat{g} \right\rangle = 2\pi \left\langle f,\, g \right\rangle. \tag{13$'$}$$

The expansion, Real-and-Imaginary-part process we just finished is an instance of *polarization.*

Dilation and translation formulas for L^2 functions

Next, our handy formulas involving translation and dilation work the same way, for any $f \in L^2$. The verification for the dilation formula is easy:

$$\begin{aligned}
(f(\lambda x))^{\wedge}_{\epsilon}(\xi) &= \int e^{-\epsilon x^2} f(\lambda x) e^{-i\xi x} dx \\
&= (1/\lambda) \int e^{-(\epsilon/\lambda^2)x^2} f(x) e^{-i(\xi/\lambda)x} dx \\
&= (1/\lambda)\widehat{f_{\epsilon/\lambda^2}}(\xi/\lambda).
\end{aligned}$$

When we make ϵ decrease to 0, we get $\mathcal{F}[f(\lambda x)](\xi) = (1/\lambda)\mathcal{F}f(\xi/\lambda)$.

The formula for the Fourier transform of a translate of f is not as easy, because we have to work with $e^{-\epsilon(x+a)^2}$ in place of $e^{-\epsilon x^2}$. Let us make a Lemma, that shows that we could have replaced our original $e^{-\epsilon x^2}$ by $e^{-\epsilon(x+c)^2}$ for any real number c.

Lemma: $\mathcal{F}f(\xi) = L^2\text{-}\lim\limits_{\epsilon \to 0} \int e^{-\epsilon x^2} f(x)\, e^{-i\xi x} dx = L^2\text{-}\lim\limits_{\epsilon \to 0} \int e^{-\epsilon(x+c)^2} f(x)\, e^{-i\xi x} dx.$

Proof: We need to show that $L^2\text{-}\lim\limits_{\epsilon \to 0} \int \left(e^{-\epsilon(x+c)^2} - e^{-\epsilon x^2}\right) f(x)\, e^{-i\xi x} dx = 0.$
We can use the "γ" argument again.

$$\begin{aligned}
&\int e^{-\gamma \xi^2} \left| \int \left(e^{-\epsilon(x+c)^2} - e^{-\epsilon x^2}\right) f(x)\, e^{-i\xi x} dx \right|^2 d\xi \\
&= \int \left(e^{-\epsilon(x+c)^2} - e^{-\epsilon x^2}\right) f(x)\, e^{-i\xi x} dx \int \left(e^{-\epsilon(y+c)^2} - e^{-\epsilon y^2}\right) \overline{f(y)} \int e^{-\gamma \xi^2}\, e^{-i\xi(x-y)}\, d\xi\, dy \\
&= 2\pi \int \left(e^{-\epsilon(x+c)^2} - e^{-\epsilon x^2}\right) f(x)\, e^{-i\xi x} dx \int \left(e^{-\epsilon(y+c)^2} - e^{-\epsilon y^2}\right) \overline{f(y)}\, \frac{e^{-(x-y)^2/4\gamma}}{\sqrt{4\pi\gamma}}\, dy \\
&= 2\pi \int \left(e^{-\epsilon(x+c)^2} - e^{-\epsilon x^2}\right) f(x) \overline{\left(\int \left(e^{-\epsilon(y+c)^2} - e^{-\epsilon y^2}\right) f(y) \frac{e^{-(x-y)^2/4\gamma}}{\sqrt{4\pi\gamma}}\, dy \right)} dx \\
&= 2\pi \left\langle \left(e^{-\epsilon(x+c)^2} - e^{-\epsilon x^2}\right) f(x), \int \left(e^{-\epsilon(y+c)^2} - e^{-\epsilon y^2}\right) f(y) \frac{e^{-(x-y)^2/4\gamma}}{\sqrt{4\pi\gamma}}\, dy \right\rangle \\
&\to 2\pi \left\langle \left(e^{-\epsilon(x+c)^2} - e^{-\epsilon x^2}\right) f(x), \left(e^{-\epsilon(x+c)^2} - e^{-\epsilon x^2}\right) f(x) \right\rangle \\
&= 2\pi \int \left(e^{-\epsilon(x+c)^2} - e^{-\epsilon x^2}\right)^2 |f(x)|^2\, dx \to 0 \quad \text{as first } \gamma, \text{ then } \epsilon, \text{ tend to } 0.
\end{aligned}$$

The last step is true by Dominated Convergence, because the squared term involving the exponentials in the last integral is at most 1, so we can use $2\pi|f(x)|^2$ as the bounding function $g(x)$, and because the integrand tends to zero a.e. as $\gamma \to 0$. This completes the proof of the Lemma.

We can now verify the translation formula for the L^2 version of the Fourier transform.

$$[\mathcal{F}(f(x-c)](\xi) = L^2\text{-}\lim_{\epsilon\to 0} \int e^{-\epsilon x^2} f(x-c)\, e^{-i\xi x} dx = L^2\text{-}\lim_{\epsilon\to 0} \int e^{-\epsilon(x+c)^2} f(x)\, e^{-i\xi(x+c)} dx = \mathcal{F}(\xi)\, e^{-i\xi c},$$

because $e^{-i\xi(x+c)} = e^{-i\xi x} e^{-i\xi c}$ so we can factor $e^{-i\xi c}$ out of the integral, and apply the Lemma.

"Moving the hat" and the Fourier Inversion formula

"Moving the hat" refers to a formula for Fourier transforms of integrable functions. If f and g are integrable then

$$(L^1 \text{ Fourier Facts 11}) \qquad \int \hat{f}(\xi) g(\xi)\, d\xi = \int f(t)\hat{g}(t)\, dt \ \text{ if } \ f \in L^1 \ \text{ and } \ g \in L^1.$$

We also need a formula involving reflection and the effect of conjugation on the Fourier transform.

$$(L^1 \text{ Fourier Facts 14}) \qquad \widehat{\overline{f}}(\xi) = \overline{\widehat{f}(-\xi)} = \widetilde{\widehat{f}}(\xi) \ \text{ and } \ \widehat{\widetilde{f}}(\xi) = \overline{\widehat{f}(\xi)} \ \text{ and } \ \widehat{f(-t)}(\xi) = \hat{f}(-\xi).$$

Since we want our formula in L^2 to be an "inner product" formula, we will conjugate g in $(L^1$ Fourier Facts 11), *after* replacing both f and g by f_ϵ and g_ϵ, respectively. We can thus apply the L^1 formulas:

$$\int \widehat{f_\epsilon}(\xi)\overline{g_\epsilon(\xi)}\, d\xi = \int f_\epsilon(t)\widehat{\overline{g_\epsilon}(t)}\, dt \ \text{ since } \ f_\epsilon \in L^1 \ \text{ and } \ g_\epsilon \in L^1.$$

Now by $(L^1$ Fourier Facts 14),

$$\widehat{\overline{g_\epsilon}(t)} = \overline{\widehat{g_\epsilon}}(-t), \quad \text{so that we have}$$

$$\int \widehat{f_\epsilon}(\xi)\overline{g_\epsilon(\xi)}\, d\xi = \int f_\epsilon(t)\overline{\widehat{g_\epsilon}}(-t)\, dt = \int f_\epsilon(-t)\overline{\widehat{g_\epsilon}(t)}\, dt.$$

When we let $\epsilon \to 0$, the far left and far right members of this equation give, by convergence in L^2, the equation

$$\int \widehat{f}(\xi)\overline{g(\xi)}\, d\xi = \int f(-t)\overline{\widehat{g}(t)}\, dt.$$

A trick: What happens if we now replace f by $\hat{f}$ in this formula? We get

$$\int \widehat{\widehat{f}}(\xi)\overline{g(\xi)}\, d\xi = \int \widehat{f}(-t)\overline{\widehat{g}(t)}\, dt.$$

We use the last equation in $(L^1$ Fourier Facts 14), which is also valid in L^2 (this is actually very easy to check), and the Plancherel Formula to get

$$\int \widehat{\widehat{f}}(\xi)\overline{g(\xi)}\, d\xi = \int \widehat{f}(-t)\overline{\widehat{g}(t)}\, dt = 2\pi \int f(-\xi)\overline{g(\xi)}\, d\xi, \quad \text{or,} \quad \langle \widehat{\widehat{f}},\, g\rangle = \langle \widehat{f}(-t),\, \widehat{g}\rangle_{dt} = 2\pi\langle f(-\xi),\, g\rangle_{d\xi}.$$

Thus

$$\langle \widehat{\widehat{f}}(\xi) - 2\pi f(-\xi),\, g(\xi)\rangle_{d\xi} = 0 \ \text{ for all } \ g \in L^2.$$

It follows (by putting $g(\xi) = \widehat{\widehat{f}}(\xi) - 2\pi f(-\xi)$) that we get the **Fourier Inversion Formula:**

$$(14) \qquad \text{For all } \ f \in L^2,\ f(x) = \frac{1}{2\pi}\widehat{\widehat{f}}(-x) \ \text{ a.e., or, abusing notation, } \ f(x) = \frac{1}{2\pi}\int \hat{f}(\xi) e^{ix\xi}\, d\xi.$$