A list of Fourier Transform formulas and properties

The list that follows is derived or obtained from references (number plus 20) following the list.

(1) $$\widehat{f}(\xi) := \int f(x)e^{-i\xi x}dx, \quad \text{where} \quad \int |f(x)|\,dx < \infty.$$

(1′) *the Fourier transform of an integrable function is a bounded function.*

(2) $$(f(t-d))\widehat{\ }_{dt}(\xi) = e^{-i\xi d}\widehat{f}(\xi)$$

(2′) $$(\tau_d f)\widehat{\ }(\xi) = e^{-i\xi d}\widehat{f}(\xi), \quad \text{where} \quad \tau_d f(t) := f(t-d) \quad \text{and} \quad d \quad \text{is a real constant.}$$

(3) $$(f(\lambda t))\widehat{\ }_{dt}(\xi) = \frac{1}{\lambda}\widehat{f}\left(\frac{\xi}{\lambda}\right)$$

(3′) $$(S_\lambda f)\widehat{\ }(\xi) = \frac{1}{\lambda}\widehat{f}\left(\frac{\xi}{\lambda}\right), \quad \text{where} \quad S_\lambda f(t) = f(\lambda t) \quad \text{and} \quad \lambda \quad \text{ia a positive constant.}$$

(4) $$(S_\lambda \tau_d f)\widehat{\ }(\xi) = \frac{1}{\lambda}e^{-i(\xi d/\lambda)}\widehat{f}\left(\frac{\xi}{\lambda}\right) = e^{-i(\xi d/\lambda)}\frac{1}{\lambda}S_{\frac{1}{\lambda}}\widehat{f}(\xi).$$

(4′) $$(f(2^j t - n))\widehat{\ }_{dt}(\xi) = e^{-in\xi/2^j}2^{-j}\widehat{f}(2^{-j}\xi).$$

(5) *If $f(t)$ is integrable, then $\widehat{f}(\xi)$ is continuous in ξ and tends to zero at infinity.*

The *convolution* of f and g is denoted $f * g$ and defined by

(6) $$f * g(t) := \int f(t-s)g(s)\,ds.$$

(7) *The convolution of two integrable functions is given by an absolutely convergent integral for a.e. t, and the convolution of two integrable functions is itself an integrable function.*

(8) $$\widehat{f*g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi).$$

(9) $$\widehat{g}(\xi) = \widehat{f'}(\xi) = i\xi\widehat{f}(\xi) \quad \text{if} \quad f \in L^1 \quad \text{and} \quad g = f' \in L^1.$$

(10) $$(tf(t))\widehat{\ }_{dt} = \int e^{-i\xi t}tf(t)\,dt = \frac{d}{d\xi}\widehat{f}(\xi) \quad \text{if} \quad f \in L^1 \quad \text{and} \quad tf(t) \in L^1.$$

(11) $$\int \widehat{f}(\xi)g(\xi)\,d\xi = \int f(t)\widehat{g}(t)\,dt \quad \text{if} \quad f \in L^1 \quad \text{and} \quad g \in L^1.$$

(12) $$f(t) = \frac{1}{2\pi}\int e^{it\xi}\widehat{f}(\xi)\,d\xi \quad \text{if} \quad f \in L^1 \quad \text{and} \quad \widehat{f} \in L^1.$$

(13) $$\widehat{\overline{f}}(\xi) = \overline{\widehat{f}(-\xi)} = \widetilde{\widehat{f}}(\xi) \quad\text{and}\quad \widehat{\widetilde{f}}(\xi) = \overline{\widehat{f}(\xi)} \quad\text{and}\quad \widehat{f(-t)}(\xi) = \widehat{f}(-\xi).$$

(14) $$\int \widehat{f}(\xi)g(\xi)e^{-i\xi\mathbf{x}}\,d\xi = \int f(t-\mathbf{x})\widehat{g}(t)\,dt \quad\text{if}\quad f\in L^1 \quad\text{and}\quad g\in L^1.$$

Derivations and sources

Foreword: a note on the integrals used here

A function $f(t)$ is *integrable* if $f(t)$ is measurable and if $\int |f(x)|\,dx < \infty$. The integral meant here is a Lebesgue integral. However, if the integral $\int |f(x)|\,dx$ is an improper Riemann integral, the usual one from Calculus, the value of that integral is the same as that of the Lebesgue integral of the same function. This works because the integrand is non-negative. There are examples where the improper Riemann integral exists, in which the Lebesgue integral does not exist. The example is

$$\int_0^\infty \frac{\sin t}{t}\,dt := \lim_{R\to\infty}\int_0^R \frac{\sin t}{t}\,dt = \frac{\pi}{2},$$

and $\int_0^\infty \frac{\sin t}{t}\,dt$ does not exist as a Lebesgue integral because the integral of the positive part and the integral of the negative part are both infinite. The “improper Lebesgue integral” exists of course because the Riemann and Lebesgue integrals over $[0, R]$ coincide. But nobody ever seems to talk about improper Lebesgue integrals! The details of the example are based on Zygmund, Ch. II Lemma (8.2).

We often say “$f \in L^1$” when f is integrable, and we often write $\|f\|_1$ for $\int |f(x)|\,dx$. More information, that I hope will be useful to you, about Lebesgue integral theory is in the note

A definition of the Fourier transform

The *Fourier transform* of an integrable function $f(x)$ is

(21) $$\widehat{f}(\xi) := \int f(x)e^{-i\xi x}dx, \quad\text{where}\quad \int |f(x)|\,dx < \infty.$$

By putting absolute-value bars under the integral sign,

$$\left|\widehat{f}(\xi)\right| \le \int |f(x)|\,dx = \|f\|_1,$$

(21′) *the Fourier transform of an integrable function is a bounded function.*

The Fourier transform of a translate

If $f(t)$ is integrable so is $f(t-d)$ for any fixed real d. Moreover, $\int f(t-d)\,dt = \int f(t)\,dt$. There is a formula for the Fourier transform of $f(t-d)$ in terms of the Fourier transform of $f(t)$. We calculate, using the literal change of variables $t \mapsto t+d$ in the following integral.

$$\widehat{(f(t-d))}_{dt}(\xi) = \int f(t-d)e^{-i\xi t}dt = \int f(t)e^{-i\xi(t+d)}dt = e^{-i\xi d}\int f(t)e^{-i\xi t}dt = e^{-i\xi d}\widehat{f}(\xi)$$

This gives us the **translation formula**

(22) $$\widehat{(f(t-d))}_{dt}(\xi) = e^{-i\xi d}\widehat{f}(\xi)$$

Sometimes it is convenient to treat translation as a linear operator. Then we write $\tau_d f(t)$ for $f(t-d)$, and the translation formula becomes

(22′) $$\widehat{(\tau_d f)}(\xi) = e^{-i\xi d}\widehat{f}(\xi), \quad\text{where}\quad \tau_d f(t) := f(t-d) \quad\text{and}\quad d \text{ is a real constant.}$$

We also have $\|\tau_d f\|_1 = \|f\|_1$.

The Fourier transform of a dilate

If $f(t)$ is integrable so is $f(\lambda t)$ for any fixed positive λ, and $\int f(\lambda t)\,dt = (1/\lambda)\int f(t)\,dt$. There is a formula for the Fourier transform of $f(\lambda t)$ in terms of the Fourier transform of $f(t)$. We calculate, using the literal change of variables $t \mapsto t/\lambda$ in the following integral.

$$(f(\lambda t))\widehat{\ }_{dt}(\xi) = \int f(\lambda t)e^{-i\xi t}dt = \int f(t)e^{-i(\xi/\lambda)t}dt/\lambda = \frac{1}{\lambda}\widehat{f}\left(\frac{\xi}{\lambda}\right)$$

This gives us the **dilation formula** (for the Fourier transform)

(23)
$$(f(\lambda t))\widehat{\ }_{dt}(\xi) = \frac{1}{\lambda}\widehat{f}\left(\frac{\xi}{\lambda}\right)$$

Somtimes it is convenient to treat dilation as a linear operator. Then we write $S_\lambda f(t)$ for $f(\lambda t)$, and the dilation formula becomes

(23′)
$$(S_\lambda f)\widehat{\ }(\xi) = \frac{1}{\lambda}\widehat{f}\left(\frac{\xi}{\lambda}\right), \quad \text{where } S_\lambda f(t) = f(\lambda t) \text{ and } \lambda \text{ ia a positive constant.}$$

We have $\|S_\lambda f\|_1 = (1/\lambda)\|f\|_1$.

Just for the record, let us compare $S_\lambda \tau_d f$ and $\tau_d S_\lambda f$.

To be sure we know what's going on, let's let $g(t) := f(t-d) = \tau_d f(t)$.

We then have $S_\lambda \tau_d f(t) = S_\lambda g(t) = g(\lambda t) = f(\lambda t - d)$.

On the other hand $\tau_d S_\lambda f(t) = S_\lambda f(t-d) = f(\lambda(t-d)) = f(\lambda t - \lambda d) = \tau_{\lambda d} f(\lambda t) = S_\lambda \tau_{\lambda d} f(t)$.

We will tend to use the first order of the operator products: $S_\lambda \tau_d$.

The Fourier transform of $S_\lambda \tau_d f$

(24)
$$(S_\lambda \tau_d f)\widehat{\ }(\xi) = \frac{1}{\lambda}e^{-i(\xi d/\lambda)}\widehat{f}\left(\frac{\xi}{\lambda}\right) = e^{-i(\xi d/\lambda)}\frac{1}{\lambda}S_{\frac{1}{\lambda}}\widehat{f}(\xi).$$

Here are the calculations:

$$(S_\lambda \tau_d f)\widehat{\ }(\xi) = \int f(\lambda t - d)e^{-i\xi t}dt = \int f(t-d)e^{-i(\xi/\lambda)t}dt/\lambda = \int f(t)e^{-i(\xi/\lambda)(t+d)}dt/\lambda = \frac{1}{\lambda}e^{-i(\xi d/\lambda)}\widehat{f}\left(\frac{\xi}{\lambda}\right).$$

We will use this case most often:

(24′)
$$(S_{2^j}\tau_n f)\widehat{(}\xi) = (f(2^j t - n))\widehat{\ }_{dt}(\xi) = e^{-i\xi n/2^j}2^{-j}\widehat{f}(2^{-j}\xi).$$

The Fourier transform of an integrable function is a continuous function that is zero at infinity

(25) *If $f(t)$ is integrable, then $\widehat{f}(\xi)$ is continuous in ξ and tends to zero at infinity.*

For the continuity we use the Lebesgue's Dominated Convergence Theorem, (1) in the classnotes *Lebesgue theory - an overview*.

Let $\xi_n \to \xi_o$. Then

$$\widehat{f}(\xi_n) - \widehat{f}(\xi_o) = \int (e^{i\xi_n t} - e^{i\xi_o t})f(t)\,dt.$$

The integrand is dominated pointwise by the integrable function $g(t) := 2|f(t)|$ and converges pointwise (in t) to zero as $n \to \infty$. The hypotheses of Lebesgue's Dominated Convergence Theorem are thus satisfied, so the integral of the limit, which is zero, is equal to the limit of the integrals, which is also the limit of $\widehat{f}(\xi_n) - \widehat{f}(\xi_o)$ as $n \to \infty$.

Since $\{\xi_n\}$ was an *arbitrary* sequence tending to ξ_o, $\lim_{\xi\to\xi_o} \widehat{f}(\xi) = \widehat{f}(\xi_o)$, so that continuity holds. Again, ξ_o was arbitrary, so continuity is true at every ξ.

That the Fourier transform of an integrable function has limit zero at infinity is called the *Riemann-Lebesgue Lemma*. We will use one of the Lebesgue facts here, (9) in the notes on Lebesgue theory: $\lim_{h\to 0}\int |f(x+h)-f(x)|\,dx = 0$. An interesting trick is used: find two formulas for some quantity and then average them to create a third formula. The trick depends on the identity $e^{\pm i\pi} = -1$. We suppose that $\xi \neq 0$. Then

$$\begin{aligned}
\widehat{f}(\xi) &= \int e^{-i\xi t} f(t)\,dt = -\int e^{-i\pi} e^{-i\xi t} f(t)\,dt \\
&= -\int e^{-i\xi\pi/\xi} e^{-i\xi t} f(t)\,dt \\
&= -\int e^{-i\xi(t+\pi/\xi)} f(t)\,dt \\
&= -\int e^{-i\xi t} f(t-\pi/\xi)\,dt \\
&= \frac{1}{2}\int e^{-i\xi t} f(t)\,dt - \frac{1}{2}\int e^{-i\xi t} f(t-\pi/\xi)\,dt \\
&= \frac{1}{2}\int e^{-i\xi t}(f(t) - f(t-\pi/\xi))\,dt.
\end{aligned}$$

Hence $|\widehat{f}(\xi)| \le \frac{1}{2}\left|\int e^{-i\xi t}(f(t)-f(t-\pi/\xi))\,dt\right| \le \frac{1}{2}\int |f(t)-f(t-\pi/\xi)|\,dt \to 0$ as $|\xi| \to \infty$. The second inequality here is (16) and the limit is (9) in the notes on Lebesgue theory.

Convolution and the Fourier Transform

The *convolution* of f and g is denoted $f * g$ and is given by

$$f * g(t) := \int f(t-s)g(s)\,ds. \tag{26}$$

As always, we have to ask whether the integral makes sense. It is not at all obvious but it is true that the integral is a finite Lebesgue integral for almost all t. The proof of this depends on Fubini's Theorem, (4) in the notes on Lebesgue theory.

First, we have by (16) in the notes on Lebesgue theory that.

$$|f * g(t)| \le \int |f(t-s)g(s)|\,ds.$$

Since we now have non-negative functions, the integrals on both sides of the inequality will exist, whether finite or not. By Tonelli's Theorem, (4′) in the Lebesgue notes,

$$\int |f*g(t)|\,dt \le \int\int |f(t-s)g(s)|\,ds\,dt = \int \left(\int |f(t-s)|\,dt\right)|g(s)|\,ds = \int |f(t)|\,dt \int |g(s)|\,ds < \infty.$$

By (31) in the Lebesgue notes, $\int |f(t-s)g(s)|\,ds < \infty$ a.e., so by Fubini's Theorem, (4) in the notes on Lebesgue theory, the integral defining the convolution is an absolutely convergent integral for a.e. t. What this amounts to is

(27) *The convolution of two integrable functions is given by an absolutely convergent integral for a.e.* t, *and the convolution of two integrable functions is itself an integrable function.*

All this gives a "multiplication" on L^1, namely the convolution. Convolution is associative and commutative, so L^1 becomes an *algebra* – a vector space with a multiplication that distributes over addition.

The Fourier transform transforms the convolution of f and g into the product of their Fourier transforms:

$$\widehat{(f*g)}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi). \tag{28}$$

Checking this is done using Fubini's theorem again:

$$\begin{aligned}(f*g)\widehat{\ }(\xi) &= \int e^{-i\xi t} f*g(t)\,dt = \int e^{-i\xi t}\int f(t-s)g(s)\,ds\,dt \\ &= \int\int e^{-i\xi(t-s)}f(t-s)e^{-i\xi s}g(s)\,ds\,dt \\ &= \int\left(\int e^{-i\xi s}g(s)\,ds\right)e^{-i\xi(t-s)}f(t-s)\,dt \\ &= \int\left(\int e^{-i\xi s}g(s)\,ds\right)e^{-i\xi(t)}f(t)\,dt, \quad \text{using the literal substitution } t\mapsto t+s, \\ &= \widehat{f}(\xi)\widehat{g}(\xi).\end{aligned}$$

Derivatives and the Fourier Transform

The Fundamental Theorem of Calculus has a counterpart in Lebesgue theory, but for the next item we will just assume that f is integrable, and that there is an integrable function g such that $f(t)=\int_{-\infty}^{t} g(s)\,ds$. Then f is continuous, differentiable almost everywhere and $f'(t)=g(t)$ a.e. **The Fourier Transform of the Derivative**

(29) $$\widehat{g}(\xi)=\widehat{f'}(\xi)=i\xi\widehat{f}(\xi) \ \text{ if } \ f\in L^1 \ \text{ and } \ g=f'\in L^1.$$

In other words, the Fourier transform transforms differentiation into multiplication of $\widehat{f}(\xi)$ by $i\xi$.

If $f(t)$ and $tf(t)$ are integrable then **Derivative of the Fourier Transform**

(30) $$(tf(t))\widehat{\ }_{dt}=\int e^{-i\xi t}tf(t)\,dt=-i\frac{d}{d\xi}\widehat{f}(\xi) \ \text{ if } \ f\in L^1 \ \text{ and } \ tf(t)\in L^1.$$

"Moving the hat:" an identity

If f and g are integrable then

(31) $$\int\widehat{f}(\xi)g(\xi)\,d\xi=\int f(t)\widehat{g}(t)\,dt \ \text{ if } \ f\in L^1 \ \text{ and } \ g\in L^1.$$

Both integrals make sense because the Fourier transforms are bounded, by $(1')$. The proof of the formula is an exercise in using the Fubini Theorem.

The Fourier Inversion Formula

If $f(t)$ and $\widehat{f}(\xi)$ are integrable then

(32) $$f(t)=\frac{1}{2\pi}\int e^{it\xi}\widehat{f}(\xi)\,d\xi \ \text{ if } \ f\in L^1 \ \text{ and } \ \widehat{f}\in L^1.$$

The proof of this involves an "approximation of the identity" argument that will be essential in the process of extending the Fourier transform to functions in L^2.

Some important "minor" facts and formulas

Complex conjugation, reflection and the Fourier transform

Sometimes we have to switch "bar" and "hat." Reflection shows up then. We have, if $f\in L^1$,

$$\widehat{\overline{f}}(\xi)=\int e^{-i\xi t}\overline{f}(t)\,dt=\overline{\int e^{i\xi t}f(t)\,dt}=\overline{\widehat{f}(-\xi)} \ \text{ and } \ \widehat{f(-t)}(\xi)=\int e^{-i\xi t}f(-t)\,dt=\int e^{i\xi t}f(t)\,dt=\widehat{f}(-\xi).$$

The "conjugated reflection" of $f(t)$ is $\overline{f(-t)}$, a combination that occurs often enough to cause people to want a notation for it. None has been widely adopted. If we need it here, we'll use $\widetilde{f}(t) := \overline{f(-t)}$ for it. This operation is an *involution,* an operation that is its own inverse. The same can be said for "reflection." But we notice that reflection commutes with the Fourier transform and conjugation does not. Now that we have introduced tha "tilde" operation, we have to write down what it does in conjunction with the Fourier transform. We list them all in one formula:

$$\widehat{\overline{f}}(\xi) = \overline{\widehat{f}(-\xi)} = \widetilde{\widehat{f}}(\xi) \quad \text{and} \quad \widehat{\widetilde{f}}(\xi) = \overline{\widehat{f}(\xi)} \quad \text{and} \quad f\widehat{(-t)}(\xi) = \widehat{f}(-\xi). \tag{34}$$

A formula for the Fourier transform of one kind of product

Since $\widehat{f}(\xi)e^{-i\xi\mathbf{x}} = \int f(t-\mathbf{x})e^{-i\xi t}\,dt$, we can, in the first integral below, "move the hat" and get

$$\int \widehat{f}(\xi)g(\xi)e^{-i\xi\mathbf{x}}\,d\xi = \int f(t-\mathbf{x})\widehat{g}(t)\,dt \quad \text{if} \quad f \in L^1 \quad \text{and} \quad g \in L^1. \tag{35}$$