A list of Fourier Transform formulas and properties

We suppose that $f \in L^2$, that is, $\int |f(x)|^2\,dx < \infty$.

(1) $$\hat{f}(\xi) := L^2\text{-}\lim_{\epsilon\to 0} \int e^{-\epsilon x^2} f(x) e^{-i\xi x} dx, \quad \text{where} \quad \int |f(x)|^2\,dx < \infty.$$

(1′) *the Fourier transform of a square-integrable function is a square-integrable function, and*

(1″) $$\int |\hat{f}(\xi)|^2\,d\xi = 2\pi \int |f(t)|^2\,dt \quad \text{Parseval's Formula}$$

(2) $$\widehat{(f(t-h))}_{dt}(\xi) = e^{-i\xi h}\hat{f}(\xi)$$

(2′) $$\widehat{(\tau_h f)}(\xi) = e^{-i\xi h}\hat{f}(\xi), \quad \text{where} \quad \tau_h f(t) := f(t-h) \quad \text{and} \quad h \quad \text{is a constant.}$$

(3) $$\widehat{(f(\lambda t))}_{dt}(\xi) = \frac{1}{\lambda}\hat{f}\left(\frac{\xi}{\lambda}\right)$$

(3′) $$\widehat{(S_\lambda f)}(\xi) = \frac{1}{\lambda}\hat{f}\left(\frac{\xi}{\lambda}\right), \quad \text{where} \quad S_\lambda f(t) = f(\lambda t) \quad \text{and} \quad \lambda \quad \text{ia a positive constant.}$$

(4) $$\widehat{(S_\lambda \tau_h f)}(\xi) = \frac{1}{\lambda} e^{-i(\xi h/\lambda)}\hat{f}\left(\frac{\xi}{\lambda}\right) = e^{-i(\xi h/\lambda)}\frac{1}{\lambda} S_{\frac{1}{\lambda}}\hat{f}(\xi).$$

(4′) $$\widehat{(f(2^j t - n))}_{dt}(\xi) = e^{-i\xi n/2^j} 2^{-j}\hat{f}(2^{-j}\xi).$$

(5) *If $f(t)$ is both integrable and square-integrable, then $\hat{f}(\xi) = \int f(x)e^{-i\xi x}dx$.*

((6)) $$f * g(t) := \int f(t-s)g(s)\,ds.$$

(7) *The convolution of two square-integrable functions is given by an absolutely convergent integral for every t, and the convolution of two square-integrable functions is a continuous function that tends to zero at infinity.*

(8) $$\widehat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi) \quad \text{if} \quad f \in L^2 \quad \text{and} \quad g \in L^1$$

(9) $$\hat{g}(\xi) = \widehat{f'}(\xi) = i\xi\hat{f}(\xi) \quad \text{if} \quad f \in L^2 \quad \text{and} \quad g = f' \in L^2.$$

(10) $$\widehat{(tf(t))}_{dt} = \int e^{-i\xi t} t f(t)\,dt = \frac{d}{d\xi}\hat{f}(\xi) \quad \text{if} \quad f \in L^2 \quad \text{and} \quad tf(t) \in L^2.$$

(11) $$\int \hat{f}(\xi)g(\xi)\,d\xi = \int f(t)\hat{g}(t)\,dt \quad \text{if} \quad f \in L^2 \quad \text{and} \quad g \in L^2.$$

The Fourier Inversion Formula

(12) $$f(t) = \frac{1}{2\pi}\widehat{\widehat{f}}(-t) \quad \text{if} \quad f \in L^2.$$

(13) $$\widehat{\overline{f}}(\xi) = \overline{\widehat{f}(-\xi)} = \widetilde{\widehat{f}}(\xi) \quad \text{and} \quad \widehat{\widetilde{f}}(\xi) = \overline{\widehat{f}(\xi)} \quad \text{and} \quad \widehat{f(-t)}(\xi) = \hat{f}(-\xi).$$

(14) $$\int \hat{f}(\xi)g(\xi)e^{-i\xi\mathbf{x}}\,d\xi = \int f(t-\mathbf{x})\hat{g}(t)\,dt \quad \text{if} \quad f \in L^2 \quad \text{and} \quad g \in L^2.$$

The Plancherel Formula

(15) $$\frac{1}{2\pi}\int \hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi = \int f(t)\overline{g(t)}\,dt \quad \text{if} \quad f \in L^2 \quad \text{and} \quad g \in L^2.$$

(15′) $$\frac{1}{2\pi}\langle \hat{f},\, \hat{g}\rangle = \langle f,\, g\rangle \quad \text{if} \quad f \in L^2 \quad \text{and} \quad g \in L^2.$$

(16) *A function $h(\xi)$ is the Fourier transform of some L^1 function if and only if there exist two L^2 functions f_1 and f_2 such that $h = f_1 * f_2$.*

(17) *A function $h(\xi)$ is the Fourier transform of some L^1 function if $h \in L^2(\mathbb{R})$ and $h' \in L^2(\mathbb{R})$.*

Derivations and sources The notes "Existence of L^2 Fourier Transform," also on the Web, contain detailed proofs of all the formulas listed here, except for (7)–(11). Those five can be derived from the others using the notes just mentioned, the notes "The Lebesgue Facts," and (in some cases) integrtation by parts.