Definitions and basic Theorems

This note is a very brief introduction to Fourier series, one of the two main "Fourier" tools we will need for wavelets: Fourier series and Fourier transforms. Fourier series give us a way to provide "coordinates" for functions, thinking of functions as "vectors." The context for the functions we are concerned about is the Hilbert space $L^2(0,\,2\pi)$ that consists of 2π–periodic functions defined on $\mathbb{R}$ with $\int_0^{2\pi} |f(x)|^2\,dx < \infty$. The inner product we use is

$$\langle f,\,g\rangle := \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}\,dx, \quad \text{where } f \in L^2(0,\,2\pi) \text{ and } g \in L^2(0,\,2\pi). \tag{1}$$

We have seen that $\frac{1}{2\pi}\int_0^{2\pi} |f(x)\overline{g(x)}|\,dx < \infty$ when $f \in L^2(0,\,2\pi)$ and $g \in L^2(0,\,2\pi)$, so $\langle f,\,g\rangle$ is a well-defined complex number. We then have $\|f\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\,dx$.

A very important set of elements of $L^2(0,\,2\pi)$ is the *trigonometric system,* also a two–way infinite sequence,

$$TS := \{e_n\}_{n\in\mathbb{Z}}, \quad \text{where } e_n(x) := e^{inx}, \quad -\infty < n < \infty. \tag{2}$$

It is routine to check that $\|e_n\| = 1$, all n, and that $\langle e_n,\,e_m\rangle = 0$ if $m \neq n$. Thus TS is an orthonormal set.

(3) **Theorem:** *TS is an orthonormal basis for $L^2(0,\,2\pi)$.*

Proof of this uses elementary methods but is hard.

The *Fourier series* of (corresponding to) a 2π–periodic function $f(x)$ is

$$f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx}, \quad \text{where } c_n := \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}\,dt = \langle f,\,e_n\rangle\,. \tag{4}$$

(5) **Exercise:** Define $f(x) := \begin{cases} x, & \text{if } |x| \le \pi; \\ \text{periodic, period } 2\pi. \end{cases}$ Calculate the $c_n(f)$. Suggestion: Integrate by parts.

Because TS *is* an orthonormal set in an inner product space, Bessel's Inequality (see [1, (18)]) implies that $\sum_{-\infty}^{\infty} |c_n|^2 \le \|f\|^2$. Because TS *is* an orthonormal basis of a Hilbert space, Parseval's Relation (see [1, (19)]) implies that $\sum_{-\infty}^{\infty} |c_n|^2 = \|f\|^2$. Though these are Theorems, we do not state them as such because they appear already in the abstract theory covered in [1].

(6) **Exercise:** Continuing (5), compute $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$, using the Parseval Relation.

Because TS *is* an orthonormal basis for $L^2(0,\,2\pi)$, the Fourier series $\displaystyle\sum_{-\infty}^{\infty} c_n e^{inx}$ actually converges in $L^2(0,\,2\pi)$, in norm, though it may not converge for every point x! It was not until 1965 that L. Carleson proved that the Fourier series does converge a.e. to $f(x)$. The proof is extraordinarily difficult. The proof that the series converges in the $L^2(0,\,2\pi)$ sense is very easy. The only difficulty is deciding what the partial sums should be. The usual approach is to use the symmetric partial sums $S_n f(x) := \displaystyle\sum_{k=-n}^{n} c_k e^{ikx}$. Then if $m < n$,

$$\|S_n f - S_m f\|^2 = \sum_{m<|k|\le n} |c_k|^2 \le \sum_{m<|k|} |c_k|^2 \to 0 \ \text{ as } \ m = \min\{m,\,n\} \to \infty.$$

Therefore $\{S_n f\}$ is a Cauchy sequence in $L^2(0,\,2\pi)$ and so it converges in $L^2(0,\,2\pi)$. To show the sequence converges to f we show that

$$\|S_n f - f\|^2 = \sum_{n<|k|} |c_k|^2 \to 0 \ \text{ as } \ n \to \infty. \tag{7}$$

(8) **Exercise:** Continuing (5), how large should n be to force $\|S_n f - f\|^2 < .001$? Suggestion: Use the Integral Test.

Some important remarks are in order here. Bessel's Inequality and Parseval's Relation are from the abstract part of the theory. They can be proved using only the properties of inner products, and in the Parseval case, the proof also uses the completeness of Hilbert spaces (i.e., the fact that every Caucy sequence has a limit in the Hilbert space).

But proving that $L^2(0,\, 2\pi)$ is a Hilbert space and that TS is an orthonormal basis are *not* part of the abstract theory. We use specific features of $L^2(0,\, 2\pi)$, Lebesgue theory and ordinary Calculus to prove these things. Thus we did not really have to check that $S_n f \to f$ in $L^2(0,\, 2\pi)$. That information is included in the abstract theory.

The point is that we want to concentrate our efforts on the things we need to do that depend on the actual problem we are dealing with. How much is just abstract theory? How much comes easily from definitions and simple calculations? How much demands real effort? All these questions have to be dealt with when we try to "solve" and *applied* problem. Exact answers may not be possible to find, or if possible, impossibly expensive. So, what is "good enough?" Better? Optimal?

If we have signals coming in and we have to store them for later use, we might try storing a list of the values of the signal. We might try sampling the signal at "enough" points and store those values. Or we might know a very good approximation for the Fourier coefficients c_n and store some of them. When it's time to bring the "stored" signal out for later use, we will really be bringing out an *approximation* of the incoming signal. To do a really good job we need to have a variety of approaches. Our knowledge of the problem and *methods* will allow us to make some good choices of what to try.

(9) **Problem:** Continuing (8), try to make your favorite mathematical software plot the $S_n f$ that you found in Exercise (8). Your answer will be to copy and print the instructions that you typed into the software. Print or sketch the curve, if you get one. if you don't, get what you can and print *that*.

In this course we will use Fourier series that are more special than square integrable. Ours will actually be trigonometric polynomials. But a few steps are needed before we wind up with 2π–periodic functions that are finite sums of the e_n's. We will first assume that our function $m_o(x)$ is in $L^2(0,\, 2\pi)$, and then we will write

$$m_o(x) \sim \sum_{-\infty}^{\infty} h_n e^{inx} =: \sum_{-\infty}^{\infty} h(n) e^{inx}.$$

We will want $m_o(x)$ to be a continuous function, so we will make the added assumption (a very big one!) that $\sum\limits_{-\infty}^{\infty} |h(n)| < \infty$. There are lots of continuous 2π–periodic functions whose Fourier-coefficient series do not converge absolutely! But then we can actually write $m_o(x) = \sum\limits_{-\infty}^{\infty} h(n) e^{inx}$. This will make working with m_o easier. We will need to work with related series such as the Fourier series for functions such as $m_o(x+\pi)$, $\overline{m_o(x)}$ and $\overline{m_o(x+\pi)}$.

(10) **Exercise:** Given that $f \in L^2(0,\, 2\pi)$ and $f(x) \sim \sum\limits_{-\infty}^{\infty} c_n e^{inx}$, find the Fourier series for $f(-x)$, $f(x+\pi)$, $\overline{f(x)}$, $e^{ix} f(x)$, and $f(2x)$. Be sure to check that $f(2x)$ is a 2π–periodic function that is in $L^2(0,\, 2\pi)$.

(11) **Exercise:** Given that $f \in L^2(0,\, 2\pi)$ and $f(x) \sim \sum\limits_{-\infty}^{\infty} c_n e^{inx}$, let $g(x) \sim \sum\limits_{-\infty}^{\infty} c_{2n} e^{inx}$ and let $h(x) \sim \sum\limits_{-\infty}^{\infty} c_{2n+1} e^{inx}$. Express f in terms of (possibly modified!) g and h.

References

[1] *Intro. to inner product and Hilbert spaces* Math 5467, Spring 2005