A list of Fourier Transform formulas and properties

The list that follows is derived or obtained from references (in the same order) following the list.

(1) $$\hat{f}(\xi) := \int f(x)e^{-i\xi x}dx, \quad \text{where} \quad \int |f(x)|\,dx < \infty.$$

(1′) *the Fourier transform of an integrable function is a bounded function.*

(2) $$\widehat{(f(t-h))}_{dt}(\xi) = e^{-i\xi h}\hat{f}(\xi)$$

(2′) $$\widehat{(\tau_h f)}(\xi) = e^{-i\xi h}\hat{f}(\xi), \quad \text{where} \quad \tau_h f(t) := f(t-h) \quad \text{and} \quad h \quad \text{is a constant.}$$

(3) $$\widehat{(f(\lambda t))}_{dt}(\xi) = \frac{1}{\lambda}\hat{f}\left(\frac{\xi}{\lambda}\right)$$

(3′) $$\widehat{(S_\lambda f)}(\xi) = \frac{1}{\lambda}\hat{f}\left(\frac{\xi}{\lambda}\right), \quad \text{where} \quad S_\lambda f(t) = f(\lambda t) \quad \text{and} \quad \lambda \quad \text{ia a positive constant.}$$

(4) $$\widehat{(S_\lambda \tau_h f)}(\xi) = \frac{1}{\lambda}e^{-i(\xi h/\lambda)}\hat{f}\left(\frac{\xi}{\lambda}\right) = e^{-i(\xi h/\lambda)}\frac{1}{\lambda}S_{\frac{1}{\lambda}}\hat{f}(\xi).$$

(4′) $$\widehat{(f(2^j t - n))}_{dt}(\xi) = e^{-i\xi n/2^j}2^{-j}\hat{f}(2^{-j}\xi).$$

(5) *If $f(t)$ is integrable, then $\hat{f}(\xi)$ is continuous in ξ and tends to zero at infinity.*

Derivations and sources

Foreword: a note on the integrals used here

A function $f(t)$ is *integrable* if $f(t)$ is measurable and if $\int |f(x)|\,dx < \infty$. The integral meant here is a Lebesgue integral. However, if the integral $\int |f(x)|\,dx$ is an improper Riemann integral, the usual one from Calculus, the value of that integral is the same as that of the Lebesgue integral of the same function. This works because the integrand is non-negative. There are examples where the improper Riemann integral exists, in which the Lebesgue integral does not exist. The example is

$$\int_0^\infty \frac{\sin t}{t}\,dt := \lim_{R\to\infty}\int_0^R \frac{\sin t}{t}\,dt = \frac{\pi}{2},$$

and $\int_0^\infty \frac{\sin t}{t}\,dt$ does not exist as a Lebesgue integral because the integral of the positive part and the integral of the negative part are both infinite. The "improper Lebesgue integral" exists of course because the Riemann and Lebesgue integrals over $[0, R]$ coincide. But nobody ever seems to talk about improper Lebesgue integrals! The details of the example, based on Zygmund Ch. II Lemma (8.2) will be an appendix to this handout.

We often say "$f \in L^1$" when f is integrable, and we often write $\|f\|_1$ for $\int |f(x)|\,dx$. More information, that I hope will be useful to you, about Lebesgue integral theory is in the note

A definition of the Fourier transform

The *Fourier transform* of an integrable function $f(x)$ is

(1) $$\hat{f}(\xi) := \int f(x)e^{-i\xi x}dx, \quad \text{where} \quad \int |f(x)|\,dx < \infty.$$

By putting absolute-value bars under the integral sign,

$$\left|\hat{f}(\xi)\right| \le \int |f(x)|\,dx = \|f\|_1,$$

(1′) *the Fourier transform of an integrable function is a bounded function.*

The Fourier transform of a translate

If $f(t)$ is integrable so is $f(t-h)$ for any fixed h. Moreover, $\int f(t-h)\,dt = \int f(t)\,dt$. There is a formula for the Fourier transform of $f(t-h)$ in terms of the Fourier transform of $f(t)$. We calculate, using the literal change of variables $t \to t+h$ in the following integral.

$$\widehat{(f(t-h))}_{dt}(\xi) = \int f(t-h)e^{-i\xi t}dt = \int f(t)e^{-i\xi(t+h)}dt = e^{-i\xi h}\int f(t)e^{-i\xi t}dt = e^{-i\xi h}\hat{f}(\xi)$$

This gives us the **translation formula**

(2) $$\widehat{(f(t-h))}_{dt}(\xi) = e^{-i\xi h}\hat{f}(\xi)$$

Sometimes it is convenient to treat translation as a linear operator. Then we write $\tau_h f(t)$ for $f(t-h)$, and the translation formula becomes

(2′) $$\widehat{(\tau_h f)}(\xi) = e^{-i\xi h}\hat{f}(\xi), \quad \text{where} \quad \tau_h f(t) := f(t-h) \quad \text{and} \quad h \quad \text{is a constant.}$$

We also have $\|\tau_h f\|_1 = \|f\|_1$.

The Fourier transform of a dilate

If $f(t)$ is integrable so is $f(\lambda t)$ for any fixed positive λ, and $\int f(\lambda t)\,dt = (1/\lambda)\int f(t)\,dt$. There is a formula for the Fourier transform of $f(\lambda t)$ in terms of the Fourier transform of $f(t)$. We calculate, using the literal change of variables $t \to t/\lambda$ in the following integral.

$$\widehat{(f(\lambda t))}_{dt}(\xi) = \int f(\lambda t)e^{-i\xi t}dt = \int f(t)e^{-i(\xi/\lambda)t}dt/\lambda = \frac{1}{\lambda}\hat{f}\left(\frac{\xi}{\lambda}\right)$$

This gives us the **dilation formula** (for the Fourier transform)

(3) $$\widehat{(f(\lambda t))}_{dt}(\xi) = \frac{1}{\lambda}\hat{f}\left(\frac{\xi}{\lambda}\right)$$

Somtimes it is convenient to treat dilation as a linear operator. Then we write $S_\lambda f(t)$ for $f(\lambda t)$, and the dilation formula becomes

(3′) $$\widehat{(S_\lambda f)}(\xi) = \frac{1}{\lambda}\hat{f}\left(\frac{\xi}{\lambda}\right), \quad \text{where} \quad S_\lambda f(t) = f(\lambda t) \quad \text{and} \quad \lambda \quad \text{ia a positive constant.}$$

We have $\|S_\lambda f\|_1 = (1/\lambda)\|f\|_1$.

Just for the record, let us compare $S_\lambda \tau_h f$ and $\tau_h S_\lambda f$.

To be sure we know what's going on, let's let $g(t) := f(t-h) = \tau_h f(t)$.

We then have $S_\lambda \tau_h f(t) = S_\lambda g(t) = g(\lambda t) = f(\lambda t - h)$.

On the other hand $\tau_h S_\lambda f(t) = S_\lambda f(t-h) = f(\lambda(t-h)) = f(\lambda t - \lambda h) = \tau_{\lambda h} f(\lambda t) = S_\lambda \tau_{\lambda h} f(t)$.

We will tend to use the first order of the operator products: $S_\lambda \tau_h$.

The Fourier transform of $S_\lambda \tau_h f$

(4)
$$\widehat{(S_\lambda \tau_h f)}(\xi) = \frac{1}{\lambda} e^{-i(\xi h/\lambda)} \hat{f}\left(\frac{\xi}{\lambda}\right) = e^{-i(\xi h/\lambda)} \frac{1}{\lambda} S_{\frac{1}{\lambda}} \hat{f}(\xi).$$

Here are the calculations:

$$\widehat{(S_\lambda \tau_h f)}(\xi) = \int f(\lambda t - h) e^{-i\xi t} dt = \int f(t-h) e^{-i(\xi/\lambda)t} dt/\lambda = \int f(t) e^{-i(\xi/\lambda)(t+h)} dt/\lambda = \frac{1}{\lambda} e^{-i(\xi h/\lambda)} \hat{f}\left(\frac{\xi}{\lambda}\right).$$

We will use this case most often:

(4′)
$$\widehat{(f(2^j t - n))}_{dt}(\xi) = e^{-i\xi n/2^j} 2^{-j} \hat{f}(2^{-j}\xi).$$

The Fourier transform of an integrable function is a continuous function zero at infinity

(5) *If $f(t)$ is integrable, then $\hat{f}(\xi)$ is continuous in ξ and tends to zero at infinity.*

For the continuity we use

We will use one of the Lebesgue facts here, (9) in the handout on Lebesgue theory, $\lim_{h\to 0} \int |f(x+h) - f(x)|\, dx = 0$.