This note contains the Heine-Borel Theorem and some useful technical points, including a proof slightly different than Rudin's that $k$-cells are compact.

**Theorem (Heine-Borel)** *A set $K \subseteq \mathbb{R}^k$ is a compact set if and only if $K$ is closed and bounded.*

*Proof:* If $K$ is compact, then $K$ is closed since $\mathbb{R}^k$ is a metric space. If $K$ is not bounded, we choose (using previous ideas) $x_1$ in $K$. Since $K$ is not bounded, there exists $x_2 \in K$ such that $|x_2| > |x_1| + 1$. We can, by induction, construct a sequence $\{x_n\}$ such that $x_n \in K$ and such that $|x_{n+1}| > |x_n| + 1$, for $n = 1, 2, \ldots$ We have done the first step. If we are given $x_n$ and $x_{n+1}$ such that both are in $K$ and such that $|x_{n+1}| > |x_n| + 1$, then since $K$ is not bounded there exists $x_{n+2} \in K$ such that $|x_{n+2}| > |x_{n+1}| + 1$. This completes the induction proof that such a sequence exists.

We have shown that every infinite subset of $K$ has a limit point in $K$. By construction, $\{x_n\}$ is actually an infinite subset of $K$, so $\{x_n\}$ has a limit point $x_\omega \in K$. Since $B_{1/3}(x_\omega)$ contains infinitely many of the $x_n$, $B_{1/3}(x_\omega)$ contains an $x_m$ such that $|x_m| > |x_\omega| + 1$ (otherwise, all the $x_n$ would have to satisfy $|x_n| \le |x_\omega| + 1$, and we can show in various ways that $\{x_n\}$ is not bounded). This gives us a contradiction:

$$|x_\omega| + 1 < |x_m| < |x_\omega| + |x_m - x_\omega| < |x_\omega| + 1/3.$$

Hence $K$ is bounded.

Now we suppose that $K$ is closed and bounded. Then there is a $k$-cell $C := [a_1, b_1] \times \cdots \times [a_k, b_k]$ that contains $K$. We will prove in a Lemma that $k$-cells are compact sets. Then since $K$ is a closed subset of a compact set, $K$ is compact. This completes the proof, except for proving the Lemma:

**Lemma** *Every $k$-cell in $\mathbb{R}^k$ is a compact set.*

*Proof:* We will prove this by showing that every infinite subset of a $k$-cell has a limit point in the $k$-cell. Let us denote some arbitrary, given, $k$-cell by $C := [a_1, b_1] \times \cdots \times [a_k, b_k]$. Suppose that $E$ is an infinite subset of $C$. Thinking of $C$ as being $C_0$, we divide each of its $k$ *edges* $[a_j, b_j]$, $j = 1, \ldots, k$ in half, into two *closed* intervals. In this way we obtain $2^k$ *subcells,* and at least one of these subcells contains infinitely many points from $E$ (otherwise, $E$ would be the union of finitely many finite sets, hence finite). We choose one of the subcells that contains infinitely many points from $E$ and call it $C_1$. Having done this once we can repeat and, by induction, obtain a sequence $\{C_n\}$ of $k$-cells such that

$$C_{n+1} \subseteq C_n, \quad \text{and each } C_n \text{ contains infinitely many points from } E.$$

If we denote by $e_j = b_j - a_j \ge 0$ the $j$-th *edge-length* of $C_0$, the $j$-th edge-length of $C_n$ is $2^{-n} e_j$ (proved by induction). Finally if we denote by $I_{n,j}$ the $j$-th edge of $C_n$, we have (for each $j$) a decreasing sequence of closed intervals whose diameters approach zero. Since each $I_{n,j}$ is compact, one of the Corollaries of the Finite Intersection Property Theorem tells us that for each $j$, $1 \le j \le k$, there is a number $x_j^*$ such that

$$\bigcap_{n=1}^{\infty} I_{n,j} = \{x_j^*\}, \quad \text{and then we set } x^* := (x_1^*, \ldots, x_k^*).$$

We can now show that $x^*$ is a limit point of $E$ and that $x^* \in C$.

Given $\delta > 0$, let $d_n := 2^{-n}\sqrt{c_1^2 + \cdots + c_K^2}$ be the diameter of $C_n$. If $d_n < \delta$, and $x \in C_n$, then since $x^* \in C_n$, $|x - x^*| \le d_n < \delta$, so $C_n \subseteq B_\delta(x^*)$. Since $C_n \cap E$ is infinite, there is in $B_\delta(x^*)$ a point of $E$ different than $x^*$, so $x^*$ is a limit point of $E$. This will complete the proof that $C$ is compact and hence complete the proof of the Lemma, once we show that there is some $n$ such that $d_n < \delta$. Suppose not. Then for all positive integers $n$,

$$2^{-n} d := 2^{-n}\sqrt{c_1^2 + \cdots + c_K^2} \ge \delta, \quad \text{or} \quad 2^n \le d/\delta.$$

But since $n + 1 \le 2^n$ for all natural numbers $n$ (proved by induction), this contradicts the Archimedian Property of the real numbers. The proof is complete.