Hint for # 10 in Chapter 11.

It is useful to write out the power series for $ff^*(z)$, and read Theorem 11.7. Then ask which complex homomorphisms are psoitive linear functionals. We use Theorem 11.32 as well. Since $F(xx^*) \geq 0$ must hold for *all* x, only one example is need to show that a given F is *not* a positive functional.

Hints for Further Problem 4 (# 14 in Chapter 11).

One thing Rudin left out is the suggestion that you prove $\overline{\varphi(-x)} = \varphi(x)$.

To show that the given functional is positive we have to show that if $\xi := f + \alpha\delta$ and

$$F(\xi) = F(f + \alpha\delta) := \int f(x)\varphi(x)\,dx + \alpha\varphi(0) \ \text{ then } \ F(\xi\xi^*) \geq 0 \ \text{ for all } \ \xi \in A.$$

We have $\xi\xi^* = f * \tilde{f} + \overline{\alpha}f + \alpha\tilde{f} + |\alpha|^2\delta$. One approach now is to work with an approximate identity given by a non-negative continuous function with compact support. Example: $K(x) := \prod_{k=1}^{n} r(x_k)$, where $r(t) := \max\{0,\, 1 - |x|\}$, sometimes called the "roof" function. The approximate identity is then given by $K_\epsilon f(x) := \int K_\epsilon(x-y)f(y)\,dy$, where $K_\epsilon(x) := \epsilon^{-n}K(x/\epsilon)$, and $K_\epsilon f \to f$ in L^1 as $\epsilon \to 0$.

By suitable changes of variable and order of integration we can write

$$F(f * \tilde{f}) = \int\int f(x)\overline{f(y)}\varphi(x-y)\,dx\,dy.$$

If f is continuous and has compact support we can approximate this integral by Riemann sums, and hence show that it is non-negative.

We can approximate $F(\xi\xi^*)$ by

$$(*) \qquad \int\int (K_\epsilon f(x) + \alpha K_\epsilon(x))\overline{(K_\epsilon f(y) + \alpha K_\epsilon(y))}\varphi(x-y)\,dx\,dy,$$

which we have shown to be non-negative, and then take the limit as $\epsilon \to 0$.

Anent (d) and the comment at the end of (c), we can use an approximate identity whose Fourier transform has compact support. For example, $\frac{1}{2\pi}\hat{r}(y) = \frac{1}{2\pi}\int r(x)e^{-ixy}dx = \frac{1}{2\pi}\frac{\sin^2 y}{y^2}$ has Fourier transform $r(x)$ by the Fourier inversion theorem. Thus if we define $K(x) := \prod_{k=1}^{n} \frac{1}{2\pi}\hat{r}(x_k)$, we have $\widehat{K}(y) = \prod_{k=1}^{n} r(y_k)$, which has compact support. Then $\widehat{K}_\epsilon(y) = \prod_{k=1}^{n} r(\epsilon y_k)$, which converges pointwise to one everywhere except at ∞, as $\epsilon \to 0$. Then by (c) and continuity $F(f + \alpha\delta) = \lim_{\epsilon\to 0} F(K_\epsilon(f + \alpha\delta)) = \lim_{\epsilon\to 0}\int_\Delta \widehat{K}_\epsilon(\hat{f} + \alpha)\,d\mu = \lim_{\epsilon\to 0}\int_{\Delta\setminus\{\infty\}} .\widehat{K}_\epsilon(\hat{f} + \alpha)\,d\mu$

Hint for # 5 in Chapter 11.

Try using the spectral mapping theorem.

Hint for # 4 in Chapter 11.

Absolute convergence of the multiple power series allows us to approximate uniformly by a polynomial: bound the tail. Continuity on the closure allows good approximation by the f_r.

Hint for # 3 in Chapter 11.

Consider the collection of all maximal ideals that contain the given closed ideal. Send email if this is not enough...

Hints for Further Problem 3.

Part (a) is useful in part (b). We are given that $(\star)$ $h(f * g) = h(f)h(g)$. When we abuse notation and write $h(f) = \int h(x)f(x)\,dx$, the equation $(\star)$ can be rewritten (with the aid of Fubini's Theorem and a change of variables)

$$\int\int f(x)g(y)h(x+y)\,dx\,dy = \int\int f(x)g(y)h(x)h(y)\,dx\,dy.$$

One approach from here is to use the Lebesgue Differentiation Theorem (in $\mathbb{R}^{2n}$).

Let us define $\tau_y f$ by $\tau_y f(x) := f(x-y)$. We know from real analysis that $\tau_y f \to f$ in L^1 as $y \to 0$.

We are assuming that h is not the zero homomorphism. Therefore there exists $f_o \in L^1$ such that $h(f_o) \neq 0$. Let us define a function $h_o(y)$ by $h_o(y) := h(\tau_y f_o)/h(f_o)$. We can show that h_o is continuous. We can then use the fact that h is a homomorphism to show that for all $f \in L^1$, $h(\tau_y f) = h_o(y)h(f)$. Then we can show that $h(x) = h_o(x)$ almost everywhere.

Hints for # 17 in Chapter 10.

First show that $\sigma(x)$ can be expressed as the union of two disjoint closed sets, because $\sigma(x)$ is not connected. Then Use 10.27 and 10.28.