The Intermediate Value Theorem: *If* $f(x)$ *is continuous on* $[a, b]$, *and* $a \le x_n \le b$ *for all* n, *and there exists a real number* c *such that* $f(a) < c < f(b)$, *then there exists* x_o, *where* $a < x_o < b$, *such that* $f(x_o) = c$.

First Proof: using LUB

We will define a set S and show that $\sup S$ is an x_o such that $f(x_o) = c$. In fact, it'll be the largest such x_o in $[a, b]$.

We define $S := \{x \in [a, b] : f(x) < c\}$. The S is non-empty because $a \in S$. Also, S is bounded above by b by thr defining criterion for membership in S. Therefore, by LUB, $\sigma := \sup S$ exists as a real number.

Next we have to show that $f(\sigma) = c$. We'll show that assuming $f(\sigma) \ne c$ leads to contradictions. We'll apply the continuity of $f(x)$ again and again without further mention.

First case: $f(\sigma) > c$.

Let $\epsilon := f(\sigma) - c$. Then there exists $\delta > 0$ such that $(x \in [a, b]$ and $\sigma - x < \delta)$ implies $f(\sigma) - f(x) < \epsilon$. Let's choose $x \in S$ such that $x > \sigma - \delta$. Then $\sigma - x < \delta$, so $\epsilon > f(\sigma) - f(x) = f(\sigma) - c + c - f(x) > f(\sigma) - c = \epsilon$ because $x \in S$. This shows that $\epsilon > \epsilon$, and gives a contradiction.

Second case: $f(\sigma) < c$. Thus $\sigma < b$ (since $f(b) > c$).

We let $\epsilon := c - f(b)$. There exists $\delta > 0$ such that $(x \in [a, b]$ and $|x - \sigma| < \delta)$ implies $f(x) - f(\sigma) < \epsilon$. We will make δ smaller, if necessary, so that $\sigma + \delta \le b$. To get a contradiction we will choose x such that $\sigma < x < \sigma + \delta$, but $f(x) < c$ so $x \in S$ but $x > \sup S$. We can take $x := \sigma + \delta/2$. Then $f(x) - f(\sigma) < \epsilon = c - f(\sigma)$, so $f(x) < c$ for $x = \sigma + \delta/2$. this is what we wanted to do: find an element of S bigger than the least upper bound of S.

Thus $f(\sigma) \ne \sigma$ is impossible, so $f(\sigma) = c$. We thus take $x_o := \sigma$, and the proof is done.

Second Proof: The author's presentation involves the inductive construction of a sequence of intervals that satisfy the conditions of the Nested Intervals Theorem.

This sort of construction involves a form of Mathematical Induction we have not yet discussed.

(Under Construction)