The Intermediate Value Theorem: *If* $f(x)$ *is continuous on* $[a, b]$, *and there exists a real number* y *such that* $f(a) < y < f(b)$, *then there exists* x_o, *where* $a < x_o < b$, *such that* $f(x_o) = y$.

First Proof: using the Completeness Axiom

We will define a set S and show that $\sup S$ is an x_o such that $f(x_o) = y$. In fact, it'll be the largest such x_o in $[a, b]$.

We define $S := \{x \in [a, b] : f(x) < y\}$. The set S is non-empty because $a \in S$. Also, S is bounded above by b by the way we defined S. Therefore, by the Completeness Axiom, $\sigma := \sup S$ exists as a real number.

Next we have to show that $f(\sigma) = y$. We'll show that assuming $f(\sigma) \neq y$ leads to contradictions. We'll apply the continuity of $f(x)$ again and again without further mention.

First case: $f(\sigma) > y$.

Let $\epsilon := f(\sigma) - y$. Then $\epsilon > 0$ and there exists $\delta > 0$ such that ($x \in [a, b]$ and $\sigma - x < \delta$) implies $|f(\sigma) - f(x)| < \epsilon$. Let's choose $x \in S$ such that $x > \sigma - \delta$. Then $\sigma - x < \delta$, so $\epsilon > f(\sigma) - f(x) = f(\sigma) - y + y - f(x) > f(\sigma) - y = \epsilon$ because $x \in S$. This shows that $\epsilon > \epsilon$, and gives a contradiction.

Second case: $f(\sigma) < y$. Thus $\sigma < b$ (since $f(b) > y$).

We let $\epsilon := y - f(\sigma)$. There exists $\delta > 0$ such that ($x \in [a, b]$ and $|x - \sigma| < \delta$) implies $f(x) - f(\sigma) < \epsilon$. We will make δ smaller, if necessary, so that $\sigma + \delta \leq b$. To get a contradiction we will choose x such that $\sigma < x < \sigma + \delta$, but $f(x) < y$ so $x \in S$ but $x > \sup S$. We can take $x := \sigma + \delta/2$. Then $f(x) - f(\sigma) < \epsilon = y - f(\sigma)$, so $f(x) < y$ for $x = \sigma + \delta/2$. This is what we wanted to do: find an element of S bigger than the least upper bound of S.

Thus $f(\sigma) \neq y$ is impossible, so $f(\sigma) = y$. We thus take $x_o := \sigma$, and the proof is done.

Exercise: In the context of the IVT and the proof just given, prove that *for all* $x \in [a, b]$, *if* $f(x) = y$ *then* $x \leq x_o$. Would it be easier to prove that *for all* $x \in [a, b]$, *if* $x > x_o$ *then* $f(x) > y$?

Second Proof: We will do a "binary search," meaning that we cut $[a, b]$ into halves, and pick the half $[a', b']$ that has $f(a') < y < f(b')$. If neither half does, we are actually done! We keep doing this over and over, and the endpoints of our chosen intervals get closer together very fast. Then continuity assures us that in the limit we find the desired point x_o. To avoid "vacuous cases" let's assume that $a < b$.

The proof will be by done by defining a sequence "inductively." We really need to justify "definition by induction!" A tool for doing this (see [1] or the Note, "Inductive Sets and Recursion," on the course Web site) is called

The Recursion Theorem *Let* X *be a non-empty set, and suppose that* $x^* \in X$. *Suppose also that* $H : X \to X$ *is a function. Then there exists a unique sequence* $x : \mathbb{N} \to X$ *such that* $x_0 = x^*$ *and such that for all natural numbers* n, $x_{n+1} = H(x_n)$.

The Recursion Theorem is a special case of the Recursive Sequence Theorem, in which we had $H(x, n)$ Here we just use $H(x, n) = H(x)$, a function "independent of n". The trouble with using the Recursion Theorem is inventing the set X and the function H.

To prove the Intermediate Value Theorem (IVT) we want to start with an interval, cut it into halves, and pick the half, $[\alpha, \beta]$, on which f behaves as it did on the original interval: $f(\alpha) < y < f(\beta)$. We repeat as long as possible.

Thus, given $[\alpha, \beta] \subseteq [a, b]$ such that $f(\alpha) < y < f(\beta)$ we put $\gamma = (\alpha + \beta)/2$ and look at $[\alpha, \gamma]$ and $[\gamma, \beta]$.

By Trichotomy exactly one of $f(\gamma) < y$, $f(\gamma) = y$ and $f(\gamma) > y$ is true.

If $f(\gamma) = y$ we put $x_o = \gamma$ and we're done because $a < x_o = \gamma < b$ and $f(x_o) = y$ is what we wanted.

If $f(\gamma) < y$ we can replace $[\alpha, \beta]$ by $[\gamma, \beta]$ and have $f(\gamma) < y < f(\beta)$. Then we define $\alpha' := \gamma$ and $\beta' := \beta$. Thus $f(\alpha') < y < f(\beta')$. We have gained this: $\beta' - \alpha' = \frac{1}{2}(\beta - \alpha)$ in this case.

If, on the other hand, $f(\gamma) > y$ we set $\alpha' := \alpha$ and $\beta' := \gamma$. Then $f(\alpha') < y < f(\beta')$ and $\beta' - \alpha' = \frac{1}{2}(\beta - \alpha)$.

In every case, $\beta' - \alpha' \leq \frac{1}{2}(\beta - \alpha)$.

This gives us a function that starts with a pair (α, β) such that $a \le \alpha \le \beta \le b$, with $f(\gamma) \le y \le f(\beta)$, and we produce the pair (α', β') such that $a \le \alpha' \le \beta' \le b$, with $f(\alpha') \le y \le f(\beta')$ and $\beta' - \alpha' \le \frac{1}{2}(\beta - \alpha)$, unless $f(\alpha') = y$ or $f(\beta') = y$, which would be nice – we'd be done! It's not very likely though.

We call this function $(\alpha', \beta') = H(\alpha, \beta)$ We therefore let our set X be the set

$$X := \{(\alpha, \beta) : a \le \alpha \le \beta \le b\}$$

and we see that $H : X \to X$. We can now apply the Recursion Theorem, with $x^* := (a, b)$, and conclude that there is a unique sequence (a_n, b_n) such that $(a_0, b_0) = x^* = (a, b)$, with $(a_{n+1}, b_{n+1}) = H(a_n, b_n)$. We can prove, by ordinary induction, that $0 \le b_n - a_n \le 2^{-n}(b - a)$. This is true when $n = 0$; if true for n then we know $(a_{n+1}, b_{n+1}) = H(a_n, b_n)$ and so $0 \le b_{n+1} - a_{n+1} \le \frac{1}{2}(\beta_n - \alpha_n) \le \frac{1}{2}2^{-n}(b - a) = 2^{-n-1}(b - a)$. Hence the Nested Intervals Theorem gives us:

$$\lim_{n\to\infty} a_n \text{ and } \lim_{n\to\infty} b_n \text{ both exist, are equal, } x_o := \lim_{n\to\infty} a_n \; (= \lim_{n\to\infty} b_n) \text{ and } a < x_o < b.$$

Since we always had $f(b_n) \ge y$ and $f(a_n) \le y$, and f is continuous, we have $y \ge \lim_{n\to\infty} f(a_n) = f(x_o)$ and $y \le \lim_{n\to\infty} f(b_n) = f(x_o)$. Thus $f(x_o) = y$. This completes the proof. It would not have been so long without our digression about the Recursion Theorem.

References

[1] Halmos, P. R., *Naive Set Theory,* Springer-Verlag, 1974. ISBN 0-387-90092-6.