Here is an example of a metric on $\mathbb{R}$ that is not complete, even though it and the usual metric detrermine the same open sets. It comes from Rudin's book *Functional Analysis,* 2nd ed., Chapter 1, Exercise 12.

We define $f(x) := x/(1+|x|)$, then we put $d(x, y) := |f(x) - f(y)|$. It is immediate that $0 \le d(x, y) = d(y, x)$ and that $d(x, y) = 0$ if $x = y$. The triangle inequality is nearly immediate: for all x, y and z in $\mathbb{R}$,

$$d(x, z) = |f(x) - f(y) + f(y) - f(z)| \le |f(x) - f(y)| + |f(y) - f(z)| = d(x, y) + d(y, z).$$

To show that $d(x, y) = 0 \Rightarrow x = y$ we use a Lemma about f that we'll apply often, leaving details to you:

Lemma: *For all* x *and* y *in* $\mathbb{R}$, $|f(x)| < 1$, $x < y \iff f(x) < f(y)$ *and* $x = f(x)/(1 - |f(x)|)$.

The Lemma shows that $d(x, y) = 0 \Rightarrow x = y$. Thus $d(x, y) = |f(x) - f(y)|$ is a metric once the Lemma is proved.

Proof: By the definition, $|f(x)| < 1$ for all $x \in \mathbb{R}$. We note that f is odd $(f(-x) = -f(x))$ and that $f(x) > 0$ if $x > 0$. If $0 < x < y$ then $y = tx$, where $t > 1$, so

$$f(x) = \frac{x}{1+x} = \frac{y/t}{1+(y/t)} = \frac{y}{t+y} < \frac{y}{1+y} = f(y).$$

If $x < y < 0$ then $f(-x) > f(-y)$ so $f(x) = -f(-x) < -f(-y) = f(y)$. Finally, if $x < 0 < y$ then $f(x) < 0 < f(y)$. Next let us verify the formula for x in terms of $f(x)$. We recall that $|f(x)| < 1$ and that x and $f(x)$ have the same sign, so that $\mathrm{sgn}(x) = \mathrm{sgn}(f(x))$, where $\mathrm{sgn}(x) := 1$ when $x > 0$, $\mathrm{sgn}(x) := 0$ when $x = 0$ and $\mathrm{sgn}(x) := -1$ when $x < 0$. We thus have

$$x = f(x)(1 + |x|) = f(x) + f(x)|x| = f(x) + f(x)\mathrm{sgn}(x)x = f(x) + f(x)\mathrm{sgn}(f(x))x = f(x) + |f(x)|x$$

(1) and thus $x(1 - |f(x)|) = f(x)$, which is what we wanted to prove, since $|f(x)| < 1$.

To prove that $f(x) < f(y)$ implies that $x < y$ we again consider the three cases $0 < f(x) < f(y)$, $f(x) < f(y) < 0$ and $f(x) < 0 < f(y)$. In the first case (1) allows us to write

$$x = f(x)/(1 - f(x)) < f(y)/(1 - f(x)) < f(y)/(1 - f(y)) = y,$$

because the denominator becomes smaller yet is still positive. In the second case,

$$x = f(x)/(1 + f(x)) < f(y)/(1 + f(x)) < f(y)/(1 + f(y)) = y.$$

Here, in the second inequality the denominator becomes larger, still positive, but the numerator is negative, so the fraction becomes less negative. The third case follows because t and $f(t)$ have the same sign for all t. This completes the proof of the Lemma.

As a Corollary we note that if $|t| < 1$ then

$$f\left(\frac{t}{1-|t|}\right) = \frac{\frac{t}{1-|t|}}{1 + \frac{|t|}{1-|t|}} = t, \quad \text{so that } f \text{ is a one–to–one correspondence between } \mathbb{R} \text{ and } (-1, 1).$$

It remains to show that a set $\mathcal{O} \subseteq \mathbb{R}$ is open with respect to the usual metric ($\mathcal{O}$ open) if and only if $\mathcal{O}$ is open with respect to d ($\mathcal{O}$ d–open). As an exercise you should verify that a set $\mathcal{O}$ in a metric space is open if and only if every point x in $\mathcal{O}$ belongs to a ball $B_r(p) \subseteq \mathcal{O}$, where $r > 0$ and $p \in \mathcal{O}$.

We suppose $\mathcal{O}$ is open and that $x \in \mathcal{O}$. Then there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subseteq \mathcal{O}$. Thus if $x - \epsilon < y < x + \epsilon$, $f(x - \epsilon) < f(y) < f(x + \epsilon)$. Since $f(x) < f(x + \epsilon)$ and $f(x - \epsilon) < f(x)$, there exists $r > 0$ such that $f(x) + r < f(x + \epsilon)$ and $f(x - \epsilon) < f(x) - r$. Therefore, if $|f(y) - f(x)| < r$ (so that $y \in B^d_r(x)$), then

$$f(x - \epsilon) < f(x) - r < f(y) < f(x) + r < f(x + \epsilon), \quad \text{so by the Lemma } B^d_r(x) \subseteq (x - \epsilon, x + \epsilon) \subseteq \mathcal{O}.$$

If now $\mathcal{O}$ is d–open and $x \in \mathcal{O}$, there exists $0 < r < 1 - |f(x)|$ such that $B^d_r(x) \subseteq \mathcal{O}$. Since $B^d_r(x)$ is the set of all $y \in \mathbb{R}$ such that $f(x) - r < f(y) < f(x) + r$, the Lemma shows that $B^d_r(x) = \{y \in \mathbb{R} : f^{-1}(f(x) - r) < y < f^{-1}(f(x) + r)\}$. Thus $B^d_r(x)$ is an open interval of real numbers containing x that is, in turn, contained in $\mathcal{O}$, so $\mathcal{O}$ is open.

Let $x_n := n$. Then $d(n, m) = \frac{|n-m|}{(1+m)(1+n)} < \frac{\max\{m,n\}}{(1+m)(1+n)} < \frac{1}{1+\min\{m,n\}} \to 0$ as $\min\{m, n\} \to \infty$, so $\{x_n\}$ is Cauchy but has no d–limit in $\mathbb{R}$. Hence $(\mathbb{R}, d)$ is not a complete metric space, even though $(\mathbb{R}, |x|)$ *is* complete and the two metrics determine the same open sets in $\mathbb{R}$.