Theorem: *Suppose that $f:\mathbb{R}\to\mathbb{R}$ is an increasing function. Then the set of discontinuities of f is countable.*
Proof: Let D ennote the set of discontinuities of f. We will show that D is either frinite or denumerable (the definition of *countable*). We defined

$$f(x+0) := \inf_{t>x} f(t) \quad \text{and} \quad f(x-0) := \sup_{t<x} f(t).$$

When $t > x$, $f(t) \geq f(x)$, so $f(x+0) \geq f(x)$. Similarly, $f(x-0) \leq f(x)$. We know then that f is discontinuous at x if and only if $J(x) := f(x+0) - f(x-0) > 0$. We call $J(x)$ the "jump" f makes at x. If $J(x) = 0$, f is continuous at x.

Moreover, we notice that $f(x+0) \leq f(t)$ if $t > x$ and $f(x-0) \geq f(t)$ if $t < x$.

We now break the set D up into pieces; we define $D_1 := \{x \in \mathbb{R} : J(x) > 1\}$ and, for integers $n > 1$, we put $D_n := \{x \in \mathbb{R} : 1/(n-1) \geq J(x) > 1/n\}$. Then $D = \bigcup_{n=1}^{\infty} D_n$ so all w e have to do is show that each set D_n is countable.

Now we have to use a little trickery: we write

$$D_n = \bigcup_{k=1}^{\infty} D_n \cap [-k,\, k].$$

Next we will show that each $D_n \cap [-k,\, k]$ is finite. It follows (**2.12, Corollary**) from this that D_n is countable, and hence (same reference) that D is countable.

If $D_n \cap [-k,\, k]$ is not empty suppose it contains points $-k \leq x_1 < \ldots < x_m \leq k$. We let $x_0 := -k$ and we let $x_{m+1} := k$. For $1 \leq i \leq m$ we notice that

$$\frac{1}{n} < f(x_i+0) - f(x_i-0) = f(x_i+0) - f(x_i) + f(x_i) - f(x_i-0) \leq f(x_{i+1}) - f(x_i) + f(x_i) - f(x_{i-1}),$$

so that

$$\frac{m}{n} < \sum_{i=1}^{m} [f(x_{i+1}) - f(x_i)] + [f(x_i) - f(x_{i-1})] = [f(k) - f(x_1)] + [f(x_m) - f(-k)] \leq 2[f(k) - f(-k)].$$

Therefore, $m < 2n\,[f(k) - f(-k)]$. Hence each set $D_n \cap [-k,\, k]$ is finite, as we wished to show.

An Example

Let $E \subseteq \mathbb{R}$ be a denumerable set; we enumerate its elements x_n, $n \in \mathbb{N}$. We notice that this does not mean the elements x_n are in the same order as the natural numbers! We define the function $f(t)$ to be

$$f(t) := \sum_{x_n < t} \frac{1}{2^n} \leq 2; \quad \text{if} \quad t = x_m, \quad \text{then} \quad J(x_m) = \frac{1}{2^m}.$$

But if $x \neq x_n$ for any n, $J(x) = 0$. Thus f is an increasing function that has E as its set of discontinuities.