§0 Introduction

"What's a wavelet?" is a question that non-technical people often ask with something akin to interest. So it's fun to bring them up. Usually I say they are little waves that can be put together to make bigger ones, or varying ones. People don't ask why I would want to do that, so I don't get to say that the object is to use just a few basic pieces, stretch them infinitely many ways, and move those in infinitely many ways to produce the "wavelet system." This system can make an exact "model" of any wave I want to (see Definition 2.1 in the text). We also want to be able to specify how to approximate our wave appropriately by using a finite number of wavelets. We also want to know, given a desired degree of accuracy, which wavelets to use, and how many. We can also ask: How accurate a model can we get if we can only afford to use a certain number of wavelets?

That's the idea, without the important details. The important technical details include *scaling* and *translation* (as in location). The overall idea is that we want to build models of waves (we will now call them *signals)* that arise out of measurements. For us, such signals $f(t)$ have "finite energy," which means that $\int_{-\infty}^{\infty} |f(t)|^2\,dt < \infty$. If we think of t as time, then $f(t)$ will usually have begun at some time T_1 and and ended at another, later time T_2. But then we think of $f(t)$ as just being zero outside the interval $[T_1, T_2]$. The interval $[T_1, T_2]$ is called the *support* of f, and f we say *has compact support.* Since the support can have *any* length, we work on the whole line. But we do not always assume a signal has compact support. Sometimes we think of a signal as having grown from zero to some peak amplitude, then fading away in the distant future, always being positive. Sometimes we work with *periodic* signals that, if they are not identically zero, would have infinite energy. In this case we restrict our attention to signals that are *periodic* and that have finite energy on each interval having the length of one period of the signal; for example $\int_{-\pi}^{\pi} |f(t)|^2\,dt < \infty$, in case the period is 2π.

We will spend some time studying the collections of *all* signals with the properties just mentioned. We will need to use Lebesgue's theory of integration rather than Riemann's. Especially when we speak in general terms, we'll use the terms "function" and "signal" interchangably. Loosely speaking, there are essentially only three such collections: all the Lebesgue measurable functions $f(t)$ defined on $\mathbb{R} := (-\infty, \infty)$ that satisfy $\int_{-\infty}^{\infty} |f(t)|^2\,dt < \infty$, which we call $L^2(\mathbb{R})$, all the Lebesgue measurable functions $f(t)$ defined on $\mathbb{R} := (-\infty, \infty)$ that satisfy $f(t + 2\pi) = f(t)$ and $\int_{-\pi}^{\pi} |f(t)|^2\,dt < \infty$, which we call $L^2(0, 2\pi)$, and all the Lebesgue measurable functions $f(x)$ on the interval $[0, 1]$ that satisfy $\int_{0}^{1} |f(t)|^2\,dt < \infty$, which we call $L^2([0, 1])$. You might object that there are also functions periodic of period P for every positive P, and you would be right, but then if $f(t + P) = f(t)$ is always true, then the function $f(\lambda t)$ is periodic with period 2π for some positive λ. Thus the periodic functions make up essentially only one collection. Again you might object that we sometime want to study functions on an arbitrary interval. Again, we can transform $L^2([a, b])$ into $L^2([0, 1])$, and vice versa, using translation and scaling. In our text, the author writes L_2 instead of L^2.

(1) **Exercise** Find the value of $\lambda > 0$ that transforms a signal $f(t)$, periodic of period P, into the signal $f(\lambda t)$, periodic of period 2π, via the *scaling* transformation $f(t) \to f(\lambda t)$. Also, find the transformation $f(t) \to f(\lambda x - \mu)$ that transforms functions defined on (a, b) to functions defined on (c, d). The transformation $f(t) \to f(x - \mu)$ is called a *translation.*

The most primitive wavelet is the *Haar function* $H(x)$, defined for all real numbers x as follows:

(2)
$$H(x) := \begin{cases} 1, & \text{if } 0 < x < 1/2, \\ -1, & \text{if } 1/2 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Associated with this basic wavelet is the *orthonormal system* (of wavelets) $H_{jk}(x)$:

(3)
$$H_{jk}(x) := 2^{j/2} H(2^j x - k).$$

The factor $2^{j/2}$ is a *normalizing* factor. The factor 2^j in $2^j x$ is a *scaling factor*, and the k in $2^j x - k$ is a *translation*. We also say that j and k are, respectively, *scaling parameter* and *translation parameter*. The scaling we use here is actually by powers of 2. We usually write H instead of H_{00}.

(4) **Exercise** Sketch the graphs of $H_{jk}(x)$ for all (j, k) with j and k being $-1, 0, 1$ (independently) and in each case compute $\displaystyle\int_{-\infty}^{\infty} |H_{jk}(x)|^2 dx$, or explain why you don't need to do the details.

When we deal with a signal that we know only thru measurement, we realize that we cannot distinguish between signals that differ at only a "few" points. For example, "the" Heaviside function $H(t)$ has the value zero when $t < 0$ and the value one when $t > 0$. The value we assign to it at zero does not matter. I suppose most people say $H(0) = 1/2$. A case might be made for making $H(0) = 0$, because that would model the turning-on of an ideal switch. But the value at $t = 0$ has no effect at all on measurements, such as $\int_{-1}^{1} |H(t) - z|^2 dt$, where z is some complex number. In fact, we could change the value of $H(t)$ at every rational number, and still the *Lebesgue* integral $\int_{-1}^{1} |H(t) - z|^2 dt$ would remain the same. Most likely, if we did that, the *Riemann* integral would simply not exist! This also leads to a subtle point involving L^2! We regard functions that differ only negligibly to be the same! The meaning of "negligible" is precise: two functions $f(t)$ and $g(t)$ in any of our L^2 spaces are called "the same" if $\displaystyle\int_{-\infty}^{\infty} |f(t) - g(t)|^2\, dt = 0$ (here the endpoints have to change when we change the space). Examples: the function $f(x)$ that is zero when x is irrational and one when x is rational is the "same as" the zero function because the Lebesgue integral of this f, after squaring it, is zero. The Riemann integral of this f does not exist over any interval with positive length. Another, similar, example has $f(x) = 0$ when x is irrational, and $f(x) = 1/n$ if x is rational and $x = m/n$ when expressed in lowest terms ($n > 0$ and m and n have no common factors besides ± 1). This function is also "essentially" zero, and this one is Riemann integrable over every bounded interval, with integral zero, whether squared or not. All this comes about as a consequence of considering only signals that can only be measured over intervals of positive length. The "subtle point" should offer you no difficulty whatever, and can probably be forgotten at once!

There are important spaces $L^p(a, b)$ that you might want to know about. These consist of the Lebesgue–measurable functions $f(x)$ such that $\int_a^b |f(x)|^p dx < \infty$. Here a and b can be finite or infinite and we require that $a < b$. As mentioned for L^2, functions $f(x)$ and $g(x)$, both in $L^p(a, b)$, that have $\int_a^b |f(t) - g(t)|^p dx = 0$ are treated as being the same. They are said to be "equal almost everywhere." For us the important L^p spaces have $p = 1$, $p = 2$ and $p = \infty$. To say a function is in L^∞ doesn't mean we raise it to an infinite power. It means that there is a number M so that $|f(x)| \leq M$ "almost everywhere."