(III.1) Introduction

Our objective now is to see how to design low-pass filters. Our idea will be to begin with a suitable finite set of numbers, construct the low-pass filter, and see how to check whether it generates a scaling function. We will then we able to construct its corresponding wavelet very easily.

(III.2) The next phase: From "low-pass filter" to scaling function

We are going to change our point of view radically! Instead of starting with a MRA we are going to start with a sequence $\{h_n\}$ that is analagous to the sequence $\{h(n)\}$ than we had before. This time, though, we have no function φ, we only have the numbers h_n. Our mission is to use the h_n to *construct* a candidate $\Phi(t)$ for a scaling function, and then show that the candidate is indeed a scaling function for a MRA.

We will make assumptions about these numbers, the main assumption being that there are only finitely many numbers in the sequence $\{h(n)\}$ that are non-zero.

We now start with finitely many non-zero coefficients h_n, set all other h_n equal to zero and make some assumptions. *We now assume that* $(h1)$ – $(h4)$ *are true, with* h_n *in place of* $h(n)$. We define our *"low-pass filter,"*

$$m_o(\xi) := \sum_n \frac{h_n}{\sqrt{2}} e^{-in\xi},$$

noticing that the sum is finite. Thus m_o is a differentiable function; indeed, m_o is a *trigonometric polynomial,* namely a function of the form $P(z)$, where $P(w)$ is a polynomial with (possibly) complex coefficients and $w = z = e^{i\xi}$.

Then we try to use the Cascade Formula to define our candidate function, that we will call Φ until we know that it really is a scaling function, and then we will call it φ. After a while we will need one more assumption, that we'll call $(h5)$. There are some mathematical hurdles to cross (or go around!), all successfully done by Ingrid Daubechies, and published in 1988 in *Communications in Pure and Applied Mathematics,* vol. 41, pp 909–996.

Here are the assumptions, most being equations we assume are true, now written in terms of the h_n. We may as well call them **Type III assumptions:**

$(h0)$ *The sequence* $\{h_n\}$ *has only finitely many non-zero terms;*

$(h1)$
$$\sum_{n\in\mathbb{Z}} |h_n|^2 = 1;$$

$(h2)$
$$\sum_{n\in\mathbb{Z}} h_n = \sqrt{2};$$

$(h3)$
$$\sum_{n\in\mathbb{Z}} (-1)^n h_n = 0;$$

$(h4)$
$$\sum_{n\in\mathbb{Z}} h_n \overline{h_{n-2k}} = 0 \quad \textit{if} \quad k \neq 0.$$

What can we verify, given that $(h1)$ – $(h4)$ are true? The linear equations $(h2)$ and $(h3)$ allow us to say that $m_o(0) = 1$ and $m_o(\pi) = 0$.

We deduced $(h4)$ from the equation $|m_o(\xi)|^2| + |m_o(\xi+\pi)|^2 = 1$ a.e., in the MRA context. We now must deduce that the equation is true everywhere from $(h4)$.

(III.3) **Theorem:** *If* $m_o(\xi) := \sum_n \frac{h_n}{\sqrt{2}} e^{-in\xi}$ *and* $(h0)$ – $(h4)$ *are true, with* h_n *in place of* $h(n)$, *then*

(*Low-Pass Identity*) $$|m_o(\xi)|^2| + |m_o(\xi+\pi)|^2 \equiv 1.$$

In particular, $|m_o(\xi)| \le 1$ *everywhere, and so for all* ξ *the vector* $(m_o(\xi), m_o(\xi+\pi)) \in \mathbb{C}^2$ *is a unit vector.*

Proof: Here are the calculations:

$$\begin{aligned}|m_o(\xi)|^2 &= \sum_n \frac{h_n}{\sqrt{2}} e^{-in\xi} \sum_m \overline{\frac{h_m}{\sqrt{2}}} e^{im\xi} \\ &= \sum_n \sum_m \frac{h_n \overline{h_m}}{2} e^{-i(n-m)\xi} \\ &= \sum_k \left(\sum_{n-m=k} \frac{h_n \overline{h_m}}{2} \right) e^{-ik\xi} \\ &= \sum_k \left(\sum_n \frac{h_n \overline{h_{n-k}}}{2} \right) e^{-ik\xi}.\end{aligned}$$

Then

$$\begin{aligned}|m_o(\xi)|^2 + |m_o(\xi+\pi)|^2 &= \sum_k \left(\sum_n \frac{h_n \overline{h_{n-k}}}{2} \right) (e^{-ik\xi} + e^{-ik(\xi+\pi)}) \\ &= \frac{1}{2} \sum_k \left(\sum_n h_n \overline{h_{n-k}} \right) e^{-ik\xi}(1 + e^{-ik\pi}) \\ &= \frac{1}{2} \sum_{k \text{ even}} \left(\sum_n h_n \overline{h_{n-k}} \right) e^{-ik\xi}(2) \\ &= \sum_k \left(\sum_n h_n \overline{h_{n-2k}} \right) e^{-i2k\xi} = 1 \text{ by } (h4) \text{ and } (h1).\end{aligned}$$

So far, all is well. It is time to try the Cascade Formula. It is really the infinite product

$$\prod_{k=1}^{\infty} m_o(\xi/2^k) \text{ that we want to work on.}$$

First question: convergence

We will show not only convergence, but that the limit is continuous. Just as, in a series, we mean by "convergence of $\sum_{n=1}^{\infty} a_n$" that the sequence of *partial sums* $s_N = \sum_{n=1}^{N} a_n$ converges as $N \to \infty$, we mean by "convergence of $\prod_{n=1}^{\infty} a_n$" that the sequence of *partial products* $p_N = \prod_{n=1}^{N} a_n$ converges as $N \to \infty$. And since we do not know in advance what the limiting value is, we use the *Cauchy convergence Criterion,* which in this case is

$$\{p_N\} \text{ converges as } N \to \infty \text{ if and only if } \lim_{\min(M,N)\to\infty} |p_M - p_N| = 0.$$

We now show that the limit exists, by using the Cauchy Criterion. We want a way to estimate how large the difference is between the partial products p_M and p_N. Let us suppose that $N > M$. Then

$$p_N - p_M = p_M \left(\prod_{k=M+1}^{N} m_o(\xi/2^k) - 1 \right).$$

First we note that $|m_o(\xi)|^2 + |m_o(\xi+\pi)|^2 = 1 \Rightarrow |m_o(\xi)| \le 1$, all ξ. Hence $|p_M(\xi)| \le 1$ because $p_M(\xi)$ is a product of $m_o(\xi/2^k)$'s. Since $m_o(0) = 1$, as k increases, $\xi/2^k$ gets close to zero rapidly. We can see how close by using the Mean Value Theorem to estimate the difference between 1 and $m_o(\xi/2^k)$. We have to be careful; the Mean Value Theorem as an *equality* is not true for complex-valued (or vector-valued) functions! But there is an *inequality* that *is* true and quite useful:

$$\|v(t) - v(t_o)\| \le \sup_s \|v'(s)\| |t - t_o|,$$

where "$\sup\limits_s$" means (here) to find the least possible upper bound on $\|v'(s)\|$ that works for all s between t and t_o. In our application, the "vector" is a complex number, so the norm is just its absolute value.

We will use as our bound the best upper bound on $|m_o'(\xi)|$ that works for *all* ξ. We don't need to know what it is; we'll even accept a larger number instead of the best bound, if it is easier to find (more in Appendix A1).

Since the low-pass filter is a trigonometric polynomial, it is differentiable, and its derivative is continuous (and is also a trigonometric polynomial). Therefore, there exists a number B such that $|m_o'(\xi)| \le B$ for all ξ. See Appendix A1 for more details. This gives us the estimate

$$|m_o(\xi/2^k) - 1| = |m_o(\xi/2^k) - m_o(0)| \le B|\xi|/2^k.$$

This means that $m_o(\xi/2^k) = 1 + \theta B|\xi|/2^k$, where θ is a complex number whose absolute value is at most 1. This is important when we want to compare $m_o(\xi/2^k)$ to 1. The thing we have to worry about is that the product could converge to zero, because $|m_o(\xi/2^k)| \le 1$. So we want the factors to stay close to 1.

We have avoided an important issue until now, though: how do we compare 1 and a *product* of complex numbers?

We'll look at $z_1 z_2 \cdots z_L - 1$ and do a lot of subtracting and adding. First, we subtract and add $z_2 \cdots z_L$ and regroup and so on, getting

$$\text{(Product Identity)}\qquad \begin{aligned} z_1 z_2 \cdots z_L - 1 &= (z_1 - 1) z_2 \cdots z_L + (z_2 \cdots z_L - 1) \\ &= (z_1 - 1) z_2 \cdots z_L + (z_2 - 1) z_3 \cdots z_L + (z_3 \cdots z_L - 1) \\ &= \vdots \\ &= (z_1 - 1) z_2 \cdots z_L + (z_2 - 1) z_3 \cdots z_L + \cdots + (z_L - 1). \end{aligned}$$

When we put our factors $m_o(\xi/2^k)$ in for z_k, the absolute value of the products of the z's are all less than or equal to 1, so we get the following estimate when we use the triangle inequality in (Product Identity):

$$|p_N - p_M| \le \sum_{k=M+1}^{N} B|\xi|/2^k < B|\xi|/2^M.$$

Now, as we let M, here the minimum of M and N, go to infinity, the Cauchy Criterion is satisfied. **The proof of convergence is done.**

proof of continuity Our estimate,

$$|p_N - p_M| < B|\xi|/2^N, \quad \text{if } N < M,$$

lets us prove that the limit, which exists everywhere, is continuous. The proof is just "locally uniform limits of continuous functions are continuous," done for this case. Although it is premature, we will now cheerfully write

$$\widehat{\Phi}(\xi) := \lim_{N\to\infty} p_N(\xi) = \prod_{k=1}^{\infty} m_o(\xi/2^k) \ \ \text{(we don't know yet that } \lim_{N\to\infty} p_N(\xi) \text{ is a Fourier transform).}$$

We let ξ_o be fixed, and suppose that $|\xi - \xi_o| \le 1$. Then $|\xi| \le |\xi_o| + 1$, so for any N, by the triangle inequality

$$\left|\widehat{\Phi}(\xi) - \widehat{\Phi}(\xi_o)\right| \le \left|\widehat{\Phi}(\xi) - p_N(\xi)\right| + |p_N(\xi) - p_N(\xi_o)| + \left|p_N(\xi_o) - \widehat{\Phi}(\xi_o)\right| \le 2B(|\xi_o|+1)/2^N + |p_N(\xi) - p_N(\xi_o)|.$$

Thus if we want $\left|\widehat{\Phi}(\xi) - \widehat{\Phi}(\xi_o)\right| < \epsilon$, we can first choose N so large that the term $2B(|\xi_o|+1)/2^N < \epsilon/2$, and then, because p_N is continuous, being a product of continuous functions, we know there exists $\delta > 0$ such that $|\xi - \xi_o| < \delta \Rightarrow |p_N(\xi) - p_N(\xi_o)| < \epsilon/2$. Hence $|\xi - \xi_o| < \delta \Rightarrow \left|\widehat{\Phi}(\xi) - \widehat{\Phi}(\xi_o)\right| < \epsilon$.

Second question: is the infinite product in L^2? Yes: Daubechies, p175 in *Ten Lectures on Wavelets.*

We'll use Fatou's Lemma here. We need to show that $\int |\widehat{\Phi}(\xi)|^2\, d\xi < \infty$. This will take several steps, and we will have to use some Lebesgue Facts. But we will also use our identity $|m_o(\xi)|^2 + |m_o(\xi+\pi)|^2 = 1$. First we define

$$F_N(\xi) = \big[\,|\xi| \le 2^N\pi\big] \prod_{k=1}^{N} |m_o(\xi/2^k)| \quad (\big[\,|\xi| \le 2^N\pi\big] = 1 \text{ if } |\xi| \le 2^N\pi, \text{ and } \big[\,|\xi| \le 2^N\pi\big] = 0 \text{ otherwise}).$$

Then F_N is a bounded function with compact support, so F_N is in L^2. We also have shown that the F_N converge pointwise to $|\widehat{\Phi}(\xi)|$. We have, after Mallat (Daubechies, loc. cit.),

$$\begin{aligned}
\int F_N(\xi)^2\, d\xi &= \int_{-2^N\pi}^{2^N\pi} \prod_{k=1}^{N} |m_o(\xi/2^k)|^2\, d\xi \\
&= \int_0^{2^{N+1}\pi} \prod_{k=1}^{N} |m_o((\xi - 2^N\pi)/2^k)|^2\, d\xi \\
&= \int_0^{2^{N+1}\pi} \prod_{k=1}^{N} |m_o((\xi/2^k) - 2^{N-k}\pi)|^2\, d\xi \\
&= \int_0^{2^{N+1}\pi} |m_o(-\pi + \xi/2^N)|^2 \prod_{k=1}^{N-1} |m_o(\xi/2^k)|^2\, d\xi \\
&= \int_0^{2^N\pi} |m_o(-\pi + \xi/2^N)|^2 \prod_{k=1}^{N-1} |m_o(\xi/2^k)|^2\, d\xi + \int_{2^N\pi}^{2^{N+1}\pi} |m_o(-\pi + \xi/2^N)|^2 \prod_{k=1}^{N-1} |m_o(\xi/2^k)|^2\, d\xi \\
&= \int_0^{2^N\pi} |m_o(-\pi + \xi/2^N)|^2 \prod_{k=1}^{N-1} |m_o(\xi/2^k)|^2\, d\xi + \int_0^{2^N\pi} |m_o(\xi/2^N)|^2 \prod_{k=1}^{N-1} |m_o(\xi/2^k)|^2\, d\xi \\
&= \int_0^{2^N\pi} \left(|m_o(-\pi + \xi/2^N)|^2 + |m_o(\xi/2^N)|^2\right) \prod_{k=1}^{N-1} |m_o(\xi/2^k)|^2\, d\xi \\
&= \int_0^{2^N\pi} \prod_{k=1}^{N-1} |m_o(\xi/2^k)|^2\, d\xi \;\; \text{[by (Low-Pass Identity)]} \;\; = \int F_{N-1}(\xi)^2\, d\xi!
\end{aligned}$$

Therefore,

(Mallat's Integral)
$$\int F_N(\xi)^2\, d\xi = \int F_1(\xi)^2\, d\xi = \int_{-2\pi}^{2\pi} |m_o(\xi/2)|^2\, d\xi = 2\int_{-\pi}^{\pi} |m_o(\xi)|^2\, d\xi = 2\pi.$$

digression: Why is $\int_{-\pi}^{\pi} |m_o(\xi)|^2\, d\xi = \pi$? We showed earlier, in the proof of (II.1), that

$$|m_o(\xi)|^2 = \frac{1}{2} \sum_k \left(\sum_n h_n \overline{h_{n-k}} \right) e^{-ik\xi}.$$

Thus

$$\int_{-\pi}^{\pi}|m_o(\xi)|^2\,d\xi=\frac{1}{2}\int_{-\pi}^{\pi}\sum_k\left(\sum_n h_n\overline{h_{n-k}}\right)e^{-ik\xi}\,d\xi=\frac{1}{2}\sum_k\int_{-\pi}^{\pi}\left(\sum_n h_n\overline{h_{n-k}}\right)e^{-ik\xi}\,d\xi=\pi,$$

because the only integral that is not zero is when $k=0$, and that one is, by $(h1)$,

$$\int_{-\pi}^{\pi}\left(\sum_n h_n\overline{h_n}\right)e^0\,d\xi=2\pi\sum_n|h_n|^2=2\pi.\ \textbf{ end of digression}$$

Fatou's Lemma comes into play in (Mallat's Integral). Since $|\widehat{\Phi}(\xi)|^2=\lim_{N\to\infty}F_N(\xi)^2=\liminf_{N\to\infty}F_N(\xi)^2$ we have

$$\int|\widehat{\Phi}(\xi)|^2\,d\xi=\int\liminf_{N\to\infty}F_N(\xi)^2\,d\xi\le\liminf_{N\to\infty}\int F_N(\xi)^2\,d\xi=2\pi,$$

and therefore $\widehat{\Phi}(\xi)\in L^2$, and so the infinite product *is* square-integrable, as claimed. We have also shown that

(Phi Fact 1) $$\frac{1}{2\pi}\int|\widehat{\Phi}(\xi)|^2\,d\xi\le 1.$$

Third question: So what?

We can use the L^2-**Fourier Inversion Formula** which for L^2 can't *really* be written as an integral. However, the first of the following equalities *is* correct, while the second is correct only sometimes:

$$\text{for all}\quad f\in L^2(\mathbb{R}),\quad\text{and a.e. } x\in\mathbb{R},\qquad f(x)=\frac{1}{2\pi}\widehat{\hat{f}}(-x)\text{``}=\text{''}\frac{1}{2\pi}\int e^{ix\xi}\hat{f}(\xi)\,d\xi.$$

This gives us a candidate for φ, although we might have to multiply by some constant:

(Phi Found) $$\Phi(t):=\frac{1}{2\pi}\widehat{\widehat{\Phi}}(-t).$$

We know now that $\Phi\in L^2$, and that $\|\Phi\|_2^2=\frac{1}{2\pi}\|\widehat{\Phi}\|_2^2\le 1$. Moreover, the L^2 Fourier transform of Φ is $\widehat{\Phi}$!

Fourth question: Does Φ have compact support? Yes

We will use this idea to show that Φ has compact support: show that whenever $f(t)$ is in L^2 and $f(t)=0$ a.e. in a certain interval (a,b), then $\langle\Phi,f\rangle=0$. This will show that $\Phi=0$ a.e. in (a,b).

We will bring in some technical tools here that are due to engineers: work with "nice" functions, so we don't have to worry about the existence of various integrals, or whether inverse Fourier transforms are given by integrals or not. This will succeed whenever the "nice" functions have appropriate density properties (for us, density in L^2).

To show that $\widehat{\Phi}$ was in L^2 we had to truncate every *partial* product, because every partial product $p_N(\xi)$ is a periodic function, and the only periodic function in $L^2(\mathbb{R})$ is the zero function. However, the partial products $p_N(\xi)$ do converge pointwise to $\widehat{\Phi}(\xi)$. We need some "nice" functions: those that are infinitely differentiable and have compact support. For more about them see the Appendix, A2. We also want to name a positive integers M_1 and M_2 such that $h_n=0$ unless $M_1\le n\le M_2$. Now we choose an infinitely differentiable function $g(t)$ with compact support, and we make sure that $g(t)=0$ whenever $t<M_1$ or $t>M_2$. Then g has the nice property that its Fourier transform is integrable and bounded and in L^2. It is also true that $g(t)=0$ in $[M_1-\epsilon,M_2+\epsilon]$, for some $\epsilon>0$ that depends on g. This g will play the rôle of the "generic" $f\in L^2$ that is zero in $(a,b)=(M_1,M_2)$. Later we'll be able to use the Fourier Inversion Formula in the form of an integral. Let's look at

$$\text{``}2\pi\langle\Phi,g\rangle=\left\langle\widehat{\Phi},g\right\rangle=\text{''}\int p_N(\xi)\overline{\hat{g}(\xi)}\,d\xi=\int\overline{\hat{g}(\xi)}\prod_{k=1}^{N}m_o(\xi/2^k)\,d\xi.$$

The part in quotes is not really there; it's where we're going, but we have to take a limit first! The product in the last integral has to be looked at in detail! For, we have here a product of sums, and we need to write it as a sum of products. What we will do is exactly that, having in mind the idea that we choose one term from each factor in the product (here there are $(M_1+M_2+1)^N$ ways to do that) and multiply them all together. We are after the ability to isolate each exponential in the sum, and see what its frequency is. Thus

$$\begin{aligned} p_N(\xi) = \prod_{k=1}^{N} m_o(\xi/2^k) = \prod_{k=1}^{N} \sum_{n_k=-M}^{M} \frac{h_{n_k}}{\sqrt{2}} e^{-i\xi n_k/2^k} &= \sum_{n_1=-M}^{M} \cdots \sum_{n_N=-M}^{M} \prod_{k=1}^{N} \frac{h_{n_k}}{\sqrt{2}} e^{-i\xi n_k/2^k} \\ &= \sum_{n_1=-M}^{M} \cdots \sum_{n_N=-M}^{M} \prod_{k=1}^{N} \frac{h_{n_k}}{\sqrt{2}} \prod_{k=1}^{N} e^{-i\xi n_k/2^k} \\ &= \sum_{n_1=-M}^{M} \cdots \sum_{n_N=-M}^{M} \left(\prod_{k=1}^{N} \frac{h_{n_k}}{\sqrt{2}} \right) e^{-i\xi \sum_{k=1}^{N} \frac{n_k}{2^k}}. \end{aligned}$$

It's the sum in the exponent that we are after. Each n_k satisfies $M_1 \le n_k \le M_2$, so if N is large enough,

$$M_1 - \epsilon < M_1 - \sum_{k=N+1}^{\infty} \frac{M_1}{2^k} = \sum_{k=1}^{N} \frac{M_1}{2^k} \le \quad T_{n_1,\ldots,n_N} := \sum_{k=1}^{N} \frac{n_k}{2^k} \quad \le \sum_{k=1}^{N} \frac{M_2}{2^k} = M_2 + \sum_{k=N+1}^{\infty} \frac{M_2}{2^k} < M_2 + \epsilon.$$

This gives us, on putting $T_{n_1,\ldots,n_N}$ in place of the sums in the exponents,

$$\begin{aligned} p_N(\xi) &= \sum_{n_1=-M}^{M} \cdots \sum_{n_N=-M}^{M} \left(\prod_{k=1}^{N} \frac{h_{n_k}}{\sqrt{2}} \right) e^{-i\xi \sum_{k=1}^{N} \frac{n_k}{2^k}} \\ &= \sum_{n_1=-M}^{M} \cdots \sum_{n_N=-M}^{M} \left(\prod_{k=1}^{N} \frac{h_{n_k}}{\sqrt{2}} \right) e^{-i\xi T_{n_1,\ldots,n_N}}. \end{aligned}$$

Next we put this monstrosity into the integral that it came from:

$$\begin{aligned} \int \overline{\hat{g}(\xi)} p_N(\xi)\, d\xi &= \int \overline{\hat{g}(\xi)} \sum_{n_1=-M}^{M} \cdots \sum_{n_N=-M}^{M} \left(\prod_{k=1}^{N} \frac{h_{n_k}}{\sqrt{2}} \right) e^{-i\xi T_{n_1,\ldots,n_N}}\, d\xi \\ &= \sum_{n_1=-M}^{M} \cdots \sum_{n_N=-M}^{M} \left(\prod_{k=1}^{N} \frac{h_{n_k}}{\sqrt{2}} \right) \left[\int \overline{\hat{g}(\xi)} e^{-i\xi T_{n_1,\ldots,n_N}}\, d\xi \right]. \end{aligned}$$

Now we are really going to simplify this! We notice that, by the Fourier Inversion Formula,

$$\frac{1}{2\pi} \int \overline{\hat{g}(\xi)} e^{-i\xi T_{n_1,\ldots,n_N}}\, d\xi = \overline{\frac{1}{2\pi} \int \hat{g}(\xi) e^{i\xi T_{n_1,\ldots,n_N}}\, d\xi} = \overline{g(T_{n_1,\ldots,n_N})},$$

and $M_1 - \epsilon < T_{n_1,\ldots,n_N} < M_2 + \epsilon$, so that $0 = g(T_{n_1,\ldots,n_N})$ because of the condition we put on the "nice" function $g(t)$, that it be zero if $M_1 - \epsilon \le t \le M_2 + \epsilon$.

Therefore, for every $g(t)$ that is infinitely differentiable, with compact support disjoint from $[M_1, M_2]$,

$$\int \overline{\hat{g}(\xi)} p_N(\xi)\, d\xi = 0.$$

Everything so far has depended on knowing that $\hat{g}$ is integrable and that p_N is bounded. Next we want to let $N \to \infty$. This time we can use Dominated Convergence. For this we take $G(\xi) := |\hat{g}(\xi)|$. Then $|\hat{g}(\xi) p_N(\xi)| \le G(\xi)$, and $\hat{g}(\xi) p_N(\xi) \to \hat{g}(\xi) \widehat{\Phi}(\xi)$. Then by the Dominated Convergence Theorem,

$$(g(-t) \perp \Phi(t)) \qquad 0 = \int \overline{\hat{g}(\xi)} p_N(\xi)\, d\xi \to \int \overline{\hat{g}(\xi)} \widehat{\Phi}(\xi)\, d\xi = \left\langle \widehat{\Phi},\, \hat{g} \right\rangle = 2\pi \left\langle \Phi,\, g \right\rangle$$

One of the useful features of our "nice" functions is that they are dense in L^2. If we require that they be zero when $M_1 \le t \le M_2$, they are no longer dense in all of L^2, but they *are* dense in the closed subspace of L^2 that consists of functions that are zero a.e. in the set where $M_1 \le t \le M_2$. Therefore a sequence $\{g_n\}$ of them can be used to approximate $[t > M_2 \text{ or } t < M_1]\Phi(t)$ in L^2. This leads to

$$\int_{t \notin [M_1,\, M_2]} |\Phi(t)|^2\, dt = \int [t > M_2 \ \text{ or } \ t < M_1]\overline{\Phi(t)}\Phi(t)\, dt = \lim_{n\to\infty} \int \overline{g_n(t)}\Phi(t)\, dt = 0,$$

so that $\Phi(t) = 0$ a.e. when $t \notin [M_1,\, M_2]$. This means that (by redefining!) Φ has compact support! Now that Φ has compact support, the fact that it's in L^2 and the Schwarz Inequality show that Φ is also in L^1. But then the Fourier transform of Φ is really given by an integral! Hence the function $\widehat{\Phi}(\xi)$ that was once a continuous limit, and then turned out to be in $L^2 \cap L^1$, can be written as

(Phi Facts 2, 3 & 4)

$$\begin{aligned} \prod_{k=1}^{\infty} m_o(\xi/2^k) = \widehat{\Phi}(\xi) &= \int \Phi(t)e^{-it\xi}\, dt \\ \text{so} \quad \widehat{\Phi}(0) &= \int \Phi(t)\, dt = 1 \\ \text{and} \quad \widehat{\Phi}(\xi) &= m_o(\xi/2)\widehat{\Phi}(\xi/2). \end{aligned}$$

It is perhaps ironic that now we have to emphasize that these last equalities hold for *every* ξ, not merely a.e.

Fifth question: Is there a constant K such that $K\Phi$ has norm 1 and has o.n. integer translates?

Here, the answer is "not necessarily, but…" We will use a new assumption of Type III that is *sufficient.*

Type III Assumption 5:

$(h5)$ $$\text{For all} \quad |\xi| \le \pi/2, \quad m_o(\xi) \ne 0.$$

In the case of the Haar scaling function, i.e. the Box function, we had $m_o(\xi) = \frac{1+e^{-i\xi}}{2} = e^{-i\xi/2}\cos\xi/2$, which is non-zero for $|\xi| < \pi$, so Type III Assumption 5 holds for the Haar scaling function.

We will now assume that we have checked that $m_o(\xi) \ne 0$ if $|\xi| \le \pi/2$. This is in addition to the other assumptions we have made. We can now show that $\{\Phi(t-n) : n \in \mathbb{Z}\}$ is an *orthogonal* set. Normality can then be guaranteed by normalizing. We will actually find that we don't need to normalize.

To show orthogonality we only have to check that, when $n \ne 0$,

$$\langle \Phi(t-n),\, \Phi(t)\rangle_{dt} = \int \Phi(t-n)\overline{\Phi(t)}\, dt = 0.$$

Let us recall the formula $\langle f,\, g\rangle = \frac{1}{2\pi}\langle \hat{f},\, \hat{g}\rangle$ and use it and the translation formula for Fourier transforms on the integral above:

$$\int \Phi(t-n)\overline{\Phi(t)}\, dt = \frac{1}{2\pi}\int e^{-in\xi}\widehat{\Phi}(\xi)\overline{\widehat{\Phi}(\xi)}\, d\xi = \frac{1}{2\pi}\int e^{-in\xi}\big|\widehat{\Phi}(\xi)\big|^2\, d\xi.$$

We previously used a *periodization argument* that we need again. We use $e^{-in(\xi+2\pi k)} = e^{-in\xi}$.

$$\begin{aligned} \frac{1}{2\pi}\int e^{-in\xi}\big|\widehat{\Phi}(\xi)\big|^2\, d\xi &= \sum_k \frac{1}{2\pi}\int_{2\pi k}^{2\pi(k+1)} e^{-in\xi}\big|\widehat{\Phi}(\xi)\big|^2\, d\xi \\ &= \sum_k \frac{1}{2\pi}\int_0^{2\pi} e^{-in\xi}\big|\widehat{\Phi}(\xi+2\pi k)\big|^2\, d\xi \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{-in\xi}\sum_k \big|\widehat{\Phi}(\xi+2\pi k)\big|^2\, d\xi. \end{aligned}$$

The interchange of sum and integral is justified by Fubini's Theorem (we recall that a sum is the integral of a piecewise constant function). To abbreviate the series we define the function

$$G(\xi) := \sum_k \left|\widehat{\Phi}(\xi + 2\pi k)\right|^2.$$

Rewriting what we started with now gives

$$(G_n) \qquad \langle \Phi(t-n),\, \Phi(t)\rangle_{dt} = \int \Phi(t-n)\overline{\Phi(t)}\, dt = \frac{1}{2\pi}\int e^{-in\xi}\left|\widehat{\Phi}(\xi)\right|^2 d\xi = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\xi} G(\xi)\, d\xi =: G_n$$

and this allows us to recognize that the numbers G_n are the Fourier coefficients of the 2π-periodic function $G(\xi)$ that is integrable over $[0,\, 2\pi]$. Our objective now is to show that all the $G_n = 0$ except when $n = 0$. In other words, we want to show that $G(\xi)$ is constant a.e. What we know so far, because the function $\Phi(t)$ has compact support, is that $G_n = 0$ if $|n| > M_2 - M_1$. For details, see the Appendix, A3. Let us define

$$\widetilde{G}(\xi) := \sum_n G_n e^{-in\xi},$$

where the sum is finite, so $\widetilde{G}(\xi)$ is a trigonometric polynomial, hence continuous, hence non-negative (not just non-negative a.e.). Here is an idea we used before, from a different perspective. We have (now true a.e.)

$$\begin{aligned}
\widetilde{G}(2\xi) &= \sum_k \left|\widehat{\Phi}(2\xi + 2\pi k)\right|^2 \\
&= \sum_k |m_o(\xi + \pi k)|^2 \left|\widehat{\Phi}(\xi + \pi k)\right|^2 \ \text{ by (Phi Fact 4)} \\
&= \sum_{k \text{ even}} |m_o(\xi)|^2 \left|\widehat{\Phi}(\xi + \pi k)\right|^2 + \sum_{k \text{ odd}} |m_o(\xi + \pi)|^2 \left|\widehat{\Phi}(\xi + \pi k)\right|^2 \\
&= |m_o(\xi)|^2 \sum_k \left|\widehat{\Phi}(\xi + 2\pi k)\right|^2 + |m_o(\xi + \pi)|^2 \sum_k \left|\widehat{\Phi}(\xi + \pi + 2\pi k)\right|^2 \\
&= |m_o(\xi)|^2 G(\xi) + |m_o(\xi + \pi)|^2 G(\xi + \pi). \\
&= |m_o(\xi)|^2 \widetilde{G}(\xi) + |m_o(\xi + \pi)|^2 \widetilde{G}(\xi + \pi).
\end{aligned}$$

The first five equalities in this string of five are true a.e. The sixth is true in this sense too at first. But since both sides of the equation

$$(G2) \qquad \widetilde{G}(2\xi) = |m_o(\xi)|^2 \widetilde{G}(\xi) + |m_o(\xi + \pi)|^2 \widetilde{G}(\xi + \pi) \ \text{ a.e.}$$

are continuous, equality a.e. changes to equality everywhere. In what follows equality everywhere will now be used. In our quest to show that $\widetilde{G}(\xi)$ is constant, it will be enough to show that the minimum and maximum values of $\widetilde{G}(\xi)$ are equal to each other.

Let us put

$$\gamma := \min_{[-\pi,\, \pi]} \widetilde{G}(\xi) \ \text{ and } \ \Gamma := \max_{[-\pi,\, \pi]} \widetilde{G}(\xi).$$

Since $\widetilde{G}(\xi)$ is continuous, $\gamma = \widetilde{G}(\xi_o)$ and $\Gamma = \widetilde{G}(\xi_1)$, where ξ_o and ξ_1 belong to $[-\pi,\, \pi]$. We want to show that 0 is a suitable choice for both ξ_o and ξ_1. But if we happen to be given non-zero values for both, we can still succeed with the help of $(G2)$, now seen to be true everywhere.

We can show by contradiction that $\gamma = \widetilde{G}(\xi_o/2)$. Thus, if we suppose that $\gamma \neq \widetilde{G}(\xi_o/2)$, then $\gamma > \widetilde{G}(\xi_o/2)$, so by $(G2)$, $(II.1)$ and **by our assumption** $(h5)$ **that** $m_o(\xi) \neq 0$ if $|\xi| \leq \pi/2$,

$$\begin{aligned}
\gamma = \widetilde{G}(2(\xi_o/2)) &= |m_o(\xi_o/2)|^2 \widetilde{G}(\xi_o/2) + |m_o((\xi_o/2) + \pi)|^2 \widetilde{G}((\xi_o/2) + \pi) \\
&> |m_o(\xi_o/2)|^2 \gamma + |m_o((\xi_o/2) + \pi)|^2 \widetilde{G}((\xi_o/2) + \pi) \\
&\geq |m_o(\xi_o/2)|^2 \gamma + |m_o((\xi_o/2) + \pi)|^2 \gamma = \gamma.
\end{aligned}$$

This says $\gamma > \gamma$, a contradiction. Therefore, $\gamma = \widetilde{G}(\xi_o/2)$. Now the argument can be repeated and then by the continuity of $\widetilde{G}(\xi)$ at 0, $\gamma = \widetilde{G}(\xi_o/2^k) \to \widetilde{G}(0)$, so $\gamma = \widetilde{G}(0)$.

Exactly the same argument, with the inequalities reversed, works to show that $\Gamma = \widetilde{G}(0)$. But then the maximum and the minimum of the continuous function $\widetilde{G}(\xi)$ coincide, so $\widetilde{G}(\xi) \equiv \widetilde{G}(0)$. Therefore, from (G_n), every $G_n = 0$ except G_0, so

$$\widetilde{G}(\xi) = \sum_n G_n e^{-in\xi} = G_0 = \widetilde{G}(0) = \frac{1}{2\pi}\int_0^{2\pi} G(\xi)\,d\xi = \frac{1}{2\pi}\int \left|\widehat{\Phi}(\xi)\right|^2 d\xi = \int \left|\Phi(t)\right|^2 dt.$$

We are nearly there: it remains to check that $\widetilde{G}(0) > 0$. But this is true because $\left|\widehat{\Phi}(\xi)\right|^2$ is continuous and, by (Phi Fact 3), $\left|\widehat{\Phi}(0)\right|^2 = 1$, so its integral cannot be zero. **We have now completed the proof that the integer translates of Φ are orthogonal, and that they can be normalized, to become orthonormal.**

Sixth question: What is the normalizing factor?

With a bit more work we can evaluate $\widetilde{G}(0) = G_0$. We will show that $\widetilde{G}(0) = 1$, so no normalization is needed, and we can rename Φ to φ and then set to work to show that φ determines a MRA.

First of all, (Phi Fact 1) shows that $(\widetilde{G}(0) =)G_0 \leq 1$.

Next, we know

$$\widetilde{G}(0) = \widetilde{G}(\xi) = \sum_k \left|\widehat{\Phi}(\xi + 2\pi k)\right|^2 \geq \sum_{|k|\leq K} \left|\widehat{\Phi}(\xi + 2\pi k)\right|^2.$$

We only know this (except for the first equality) is true a.e. But the inequality

$$\widetilde{G}(\xi) \geq \sum_{|k|\leq K} \left|\widehat{\Phi}(\xi + 2\pi k)\right|^2$$

involves only continuous functions, so it must be true everywhere. Therefore

$$\widetilde{G}(0) \geq \sum_{|k|\leq K} \left|\widehat{\Phi}(2\pi k)\right|^2 \geq \left|\widehat{\Phi}(0)\right|^2 = 1.$$

Hence, $\widetilde{G}(0) = 1$. We are ready to define our possible scaling function. But first, **a byproduct of this argument:** *If* $m \neq 0$ *then* $\widehat{\Phi}(2\pi m) = 0$. This fact can actually be shown directly from the definition of $\widehat{\Phi}(\xi)$ as an infinite product, because if $\xi = 2\pi m$, then $\xi/2^k$ will be an odd multiple of π for some k, and the partial products from there on will all be zero.

A candidate for a scaling function: $\varphi(t) := \Phi(t)$.

Seventh question: Does this φ determine a MRA? Reconstituted from Hernandez and Weiss, *A First Course on Wavelets.*

The answer is "Yes." To verify, we begin by defining V_0 to be the closure of the span of the integer translates of φ. As we know, this is the set of all functions that can be expressed as the L^2-limit of a series $\sum_n c_n\varphi(t-n)$, where it is required that $\sum_n |c_n|^2 < \infty$. This shows by construction that V_0 is a closed subspace, so that

MRA (v) holds.

Next, we define, for each integer j,

$$V_j := \{f(2^j t) : f(t) \in V_0\}.$$

If $F \in V_j$, then $F(2t) = f(2^j(2t))$, where $f \in V_0$. But then $F(2t) = f(2^{(j+1)}t)$, so by definition, $F(2t) \in V_{j+1}$. Now we let $F(t)$ be a function such that $F(2t) \in V_{j+1}$. Then by definition, $F(2t) = f(2^{(j+1)}t)$ for some $f(t) \in V_o$. But then $F(t) = f(2^j t)$ for some $f(t) \in V_o$, so $F \in V_j$. This shows that

MRA (iv) holds.

Since $\|f(2^jt)\|_{2,dt} = \frac{1}{2^{j/2}}\|f\|_2$, sequences in V_j converge if and only if the sequences in V_0 that define them converge. Since (by construction!) V_0 is closed, all the V_j are closed. This shows that

MRA (i) holds.

Let us show that the spaces V_j are nested. Phi Fact 4 says

$$\widehat{\Phi}(\xi) = m_o(\xi/2)\widehat{\Phi}(\xi/2) = \left(\sum_n \frac{h_n}{\sqrt{2}} e^{-in\xi}\right)\widehat{\Phi}(\xi/2) = \sum_n \frac{h_n}{\sqrt{2}}\left(e^{-in\xi}\widehat{\Phi}(\xi/2)\right).$$

Now we need to recognize the presence of two Fourier Transform formulas. We recall that $e^{-in\xi}\frac{1}{2}\widehat{\Phi}(\xi/2)$ is the Fourier Transform of $\Phi(2t-n)$, so $e^{-in\xi}\widehat{\Phi}(\xi/2)$ is the Fourier transform of $2\Phi(2t-n)$. When we put these back into Phi Fact 4, and replace Φ by φ and remove the Fourier transforms, we get

$$\varphi(t) = \sum_n h_n\sqrt{2}\varphi(2t-n), \quad \text{so} \quad \varphi \in V_1.$$

But then, because V_1 is a closed subspace, the closure of the span of the integer translates of φ, namely V_0, is contained in V_1. By invoking (iv) we can conclude that the sequence of spaces is nested. We have shown that

MRA (ii) holds.

To show that (iii) is true there are two things to do. Let us begin by showing that (i), (ii), (iv) and (v) show that

$$\bigcap_j V_j = \{0\}.$$

In other words, if we know that (i), (ii), (iv) and (v) hold we need not check that the intersection of all the V_j is the zero subspace.

We suppose then that $Z(t) \in \bigcap_j V_j$, i.e., $Z(t)$ is in every V_j. Our strategy will be to show that $\int_{2^j\pi}^{2^{j+1}\pi} |\widehat{Z}(\xi)|\,d\xi = 0$ for every integer j. The same argument will work to show that $\int_{-2^{j+1}\pi}^{-2^j\pi} |\widehat{Z}(\xi)|\,d\xi = 0$ and so $\widehat{Z}(\xi) = 0$ a.e. will be the result. Let us work with a large positive integer N. Then since $Z \in V_{-N}$ the function

$$f_N(t) := 2^{N/2}Z(2^Nt) \in V_0.$$

Let's be sure. By the way we defined $Z(t) \in V_N$ there exists $f \in V_0$ such that $Z(t) = f(2^{-N}t)$. But then $Z(2^Nt) = f(t) \in V_0$. Our definition $f_N(t) = 2^{N/2}Z(2^Nt)$ contains a norm-preserving factor, chosen so that $\|f_N\|_2 = \|Z\|_2$. Now we can write

$$f_N(t) = \sum_k c_{N,k}\varphi(t-k), \quad \text{with} \quad \sum_k |c_{N,k}|^2 = \|f_N\|_2^2 = \|Z\|_2^2.$$

Therefore

$$\hat{f}_N(\xi) = \left(\sum_k c_{N,k}e^{-ik\xi}\right)\widehat{\varphi}(\xi) := \pi_N(\xi)\widehat{\varphi}(\xi),$$

where $\pi_N(\xi)$ is 2π-periodic and in $L^2(\mathbb{T})$. Indeed, we know that

$$\|\pi_N\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |\pi_N(\xi)|^2\,d\xi = \sum_k |c_{N,k}|^2 = \|f_N\|_2^2 = \|Z\|_2^2.$$

Consequently, from the way we defined f_N,

$$2^{-N/2}\widehat{Z}(2^{-N}\xi) = \hat{f}_N(\xi) = \pi_N(\xi)\widehat{\varphi}(\xi).$$

We can start our strategy now, using the last equation. We assume that $j+N$ is a positive integer.

$$\int_{2^j\pi}^{2^{j+1}\pi} |\widehat{Z}(\xi)|\,d\xi = \int_{2^{j+N}\pi}^{2^{j+N+1}\pi} |\widehat{Z}(2^{-N}\xi)|2^{-N}\,d\xi = \int_{2^{j+N}\pi}^{2^{j+N+1}\pi} |\pi_N(\xi)\widehat{\varphi}(\xi)|2^{-N/2}\,d\xi.$$

Then, by the Schwarz inequality, the periodicity of $\pi_N(\xi)$, and multiplying and dividing by $2^{j/2}$,

$$\begin{aligned}\int_{2^{j+N}\pi}^{2^{j+N+1}\pi} |\pi_N(\xi)\widehat{\varphi}(\xi)|2^{-N/2}\,d\xi &\le 2^{j/2}\left(2^{-N-j}\int_{2^{j+N}\pi}^{2^{j+N+1}\pi} |\pi_N(\xi)|^2\,d\xi\right)^{1/2}\left(\int_{2^{j+N}\pi}^{2^{j+N+1}\pi} |\widehat{\varphi}(\xi)|^2\,d\xi\right)^{1/2}\\ &= 2^{j/2}\left(\int_0^{2\pi} |\pi_N(\xi)|^2\,d\xi\right)^{1/2}\left(\int_{2^{j+N}\pi}^{2^{j+N+1}\pi} |\widehat{\varphi}(\xi)|^2\,d\xi\right)^{1/2}\\ &\le 2^{j/2}\left(\int_0^{2\pi} |\pi_N(\xi)|^2\,d\xi\right)^{1/2}\left(\int_{2^{j+N}\pi}^{\infty} |\widehat{\varphi}(\xi)|^2\,d\xi\right)^{1/2}.\end{aligned}$$

The equality here came about because the interval $[2^{j+N}\pi,\, 2^{j+N+1}\pi)$ is made up of 2^{j+N} adjacent translates of the interval $[0,\, 2\pi)$, and $\pi_N(\xi)$ is periodic of period 2π. Let us clean up what we have done so far:

$$\int_{2^j\pi}^{2^{j+1}\pi} |\widehat{Z}(\xi)|\,d\xi \le 2^{j/2}\left(\int_0^{2\pi} |\pi_N(\xi)|^2\,d\xi\right)^{1/2}\left(\int_{2^{j+N}\pi}^{\infty} |\widehat{\varphi}(\xi)|^2\,d\xi\right)^{1/2}.$$

We set down earlier that $\frac{1}{2\pi}\int_0^{2\pi} |\pi_N(\xi)|^2\,d\xi = \sum_k |c_{N,k}|^2 = \|f_N\|_2^2 = \|Z\|_2^2$, so the first integral above on the right is independent of N, being equal to $\|Z\|_2/\sqrt{2\pi}$. But the second integral tends to zero as $N \to \infty$. Again the integral on the left is independent of N, hence it must be zero.

We have completed half of (iii). The other half is about the density of a subspace. We will take advantage of the compact support of φ, which guarantees that φ is integrable, and hence that $\widehat{\varphi}$ is continuous, and use Phi Fact 3, that $\widehat{\varphi}(0)=1$. We only need $\widehat{\varphi}(0)\neq 0$. We can find $\delta>0$, to save until a bit later, such that $|\xi|<\delta \Rightarrow \widehat{\varphi}(\xi)\neq 0$.

We will use the theorem that says a subspace is dense if the only vector perpendicular to everything in it is the zero vector.

We will let $W := \bigcup_j V_j$, and suppose that $g \in L^2$ and $g \perp W$. If $f \in W$ this means that $f \in V_J$ for some integer J. Then $f \in V_j$ for all $j \ge J$. Let us recall this means $f(t)=\sum_k c_k\varphi_{J,k}(t) = \sum_k c_k 2^{J/2}\varphi(2^J t-k)$, where $\sum_k |c_k|^2<\infty$. It is also true by (ii) that, for each $j>J$, $f(t)=\sum_k c_{j,k}\varphi_{j,k}(t)=\sum_k c_{j,k}2^{j/2}\varphi(2^j t-k)$, where $\sum_k |c_{j,k}|^2=\sum_k |c_k|^2<\infty$. Let us show that $f(t)\in W \Rightarrow f(t+r)\in W$ whenever $r=m/2^\ell$ is a *dyadic rational:* $m\in\mathbb{Z}$ and $\ell\ge 0$. We can write $r=(2^{j-\ell}m)/2^j$ if $0\le \ell\le j$. We use the second type of expression for f, and we choose $j\ge \max\{J,\,\ell\}$:

$$f(t+r)=\sum_k c_{j,k}2^{j/2}\varphi(2^j(t+r)-k)=\sum_k c_{j,k}2^{j/2}\varphi(2^j t-(k-2^{j-\ell}m))=\sum_k c_{j,k+2^{j-\ell}m}2^{j/2}\varphi(2^j t-k)\in V_j\subseteq W.$$

Now we bring g back: since $g\perp W$, for every $f\in W$ and every dyadic rational r,

$$0=\langle f(t+r),\,g\rangle_{dt}=\int f(t+r)\overline{g(t)}\,dt.$$

Let's define a function

$$h(x) := \int f(t+x)\overline{g(t)}\,dt.$$

We can use one of the Lebesgue facts to show that $h(x)$ is a continuous function of x. We have

$$|h(x)-h(x')| \le \int |f(t+x)-f(t+x')||g(t)|\,dt \le \left(\int |f(t+x)-f(t+x')|^2\,dt\right)^{1/2}\left(\int |g(t)|^2\,dt\right)^{1/2}$$

by the Schwarz inequality. We now recall Lebesgue Fact (9),

Continuity of norms with respect to translation
If $\left\{\int |f(x)|^p\,dx\right\}^{1/p} < \infty$, where $1 \le p < \infty$, then $\lim_{h\to 0}\left\{\int |f(x+h)-f(x)|^p\,dx\right\}^{1/p} = 0$.

We apply this with $p = 2$ and $h = x - x'$, which we can arrange by the literal change of variable $t \to t - x'$. Hence h is continuous.

Since every real number x is the limit of a sequence of dyadic rationals r, we have shown that $h(x) \equiv 0$. Once more we recall the formula $\langle f,\, g\rangle = \frac{1}{2\pi}\langle \hat{f},\, \hat{g}\rangle$, which gives

$$h(x) = 0 = \int f(t+x)\overline{g(t)}\,dt = \langle f(t+x),\, g\rangle_{dt} = \frac{1}{2\pi}\int e^{ix\xi}\hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi.$$

The function $\hat{f}(\xi)\overline{\hat{g}(\xi)}$ is integrable and has Fourier transform identically zero. It is a Fourier Fact that this means $\hat{f}(\xi)\overline{\hat{g}(\xi)} = 0$ a.e.

Now we are ready for the last step: we choose $f(t) = 2^j\varphi(2^j t)$. Then $f \in W$, and $\hat{f}(\xi) = \widehat{\varphi}(2^{-j}\xi)$, so

$$\widehat{\varphi}(2^{-j}\xi)\overline{\hat{g}(\xi)} = 0 \quad \text{a.e.}$$

We selected $\delta > 0$ near the start of this discussion such that $|\xi| < \delta \Rightarrow \widehat{\varphi}(\xi) \ne 0$. Therefore here, for a.e. $2^{-j}|\xi| < \delta$ we have $\overline{\hat{g}(\xi)} = 0$. This means $\hat{g}(\xi) = 0$ a.e. in $|\xi| < 2^j\delta$. As we let $j \to \infty$, we conclude that $\hat{g}(\xi) = 0$ a.e. Hence, by Fourier Inversion, $g(t) = 0$ a.e. also. We have shown that

$$\overline{\bigcup_j V_j} = L^2(\mathbb{R}),$$

and hence that

MRA (iii) holds. This completes the proof that φ determines a MRA.

We began in (III.2), worked with finitely many non-zero h_n's, that satisfied $(h0)$–$(h5)$ and eventually arrived at a scaling function φ with compact support that determined a MRA.

Let us summarize, beginning on the next page with a statement of what we have done, in the form of a Theorem.

(III.4) Theorem

Suppose we can find a sequence $\{h_n\}$ *of complex numbers with all the six properties that follow.*

$(h0)$ *The sequence* $\{h_n\}$ *has only finitely many non-zero terms.*

$(h1)$
$$\sum_n |h_n|^2 = 1.$$

$(h2)$
$$\sum_n h_n = \sqrt{2}.$$

$(h3)$
$$\sum_n (-1)^n h_n = 0.$$

$(h4)$
$$\sum_n h_n \overline{h_{n-2k}} = \delta_{k0} = \begin{cases} 1, & \text{if } k = 0; \\ 0, & \text{if } k \neq 0. \end{cases}$$

We then define the trigonometric polynomial

$$m_o(\xi) := \sum_n \frac{h_n}{\sqrt{2}} e^{-in\xi},$$

and suppose that

$(h5)$
$$|\xi| \leq \pi/2 \Rightarrow m_o(\xi) \neq 0.$$

Then the function

$$\widehat{\varphi}(\xi) := \prod_{k=1}^{\infty} m_o(\xi/2^k)$$

defines, by taking its inverse Fourier transform, a function φ *that has compact support and satisfies all the conditions of a MRA when we define the subspace* V_0 *to be the closure of the span of the integer translates of* φ, *and define, for each integer* j,

$$V_j := \{f(2^j t) : f(t) \in V_0\}.$$

A leftover question: How do we find such sets of numbers? In particular, how many k have to be checked to ensure that $(h4)$ is true?

OUTLINE of "From "low-pass filter" to scaling function"

We started with finitely many non-zero coefficients h_n (with $h_n = 0$ unless $M_1 \leq n \leq M_2$ even though *some* h_n with $M_1 \leq n \leq M_2$ might be zero). We made some assumptions about them and about the low-pass filter they determine. We assumed that $(h1)$ – $(h4)$ are true, with h_n in place of $h(n)$ (see the Theorem above). We defined the trigonometric polynomial

$$m_o(\xi) := \sum_n \frac{h_n}{\sqrt{2}} e^{-in\xi}.$$

The linear equations $(h2)$ and $(h3)$ allowed us to say that $m_o(0) = 1$ and $m_o(\pi) = 0$.

Theorem: *If* $m_o(\xi) := \sum_n \frac{h_n}{\sqrt{2}} e^{-in\xi}$ *and* $(h1)$ – $(h4)$ *are true, with* h_n *in place of* $h(n)$, *then*

$$|m_o(\xi)|^2| + |m_o(\xi+\pi)|^2 \equiv 1.$$

In particular, $|m_o(\xi)| \le 1$ *everywhere, and so for all* ξ *the vector* $(m_o(\xi),\, m_o(\xi+\pi)) \in \mathbb{C}^2$ *is a unit vector.*

We used the Cascade Formula to define a candidate for φ. It was really the infinite product

$$\widehat{\Phi}(\xi) := \prod_{k=1}^{\infty} m_o(\xi/2^k)$$

that we worked with. The notation $\widehat{\Phi}(\xi)$ is not yet justified.

First question: convergence

We showed not only convergence, but that the limit is continuous.
We showed that the limit exists by using the Cauchy Criterion.

Second question: is the infinite product in L^2? Yes: Daubechies, p175 in *Ten Lectures on Wavelets.*

An argument of Mallat was used, and Fatou’s Lemma, to show that $\int |\widehat{\Phi}(\xi)|^2\, d\xi < \infty$. We also used the identity $|m_o(\xi)|^2 + |m_o(\xi+\pi)|^2 = 1$. We showed that $\|\widehat{\Phi}(\xi)\|^2 \le 2\pi$.

Third question: So what?

We used the L^2**-Fourier Inversion Formula:**

$$\text{for all}\quad f \in L^2(\mathbb{R}),\quad \text{and a.e. } x \in \mathbb{R}, \qquad f(x) = \frac{1}{2\pi}\hat{\hat{f}}(-x) \text{ “ = ” } \frac{1}{2\pi}\int e^{ix\xi}\hat{f}(\xi)\, d\xi.$$

This gave us a candidate for φ, although we might have to multiply by some constant:

(Phi Found) $$\Phi(t) := \frac{1}{2\pi}\hat{\widehat{\Phi}}(-t).$$

We found that $\Phi \in L^2$, and that $\|\Phi\|_2^2 = \frac{1}{2\pi}\|\widehat{\Phi}\|_2^2 \le 1$.

Fourth question: Does Φ have compact support? Yes

This used some very “nice” functions, the ones that are infinitely differentiable and have compact support. Their properties (will be) discussed in the Fourier Facts notes. We looked at

$$\int \overline{\hat{g}(\xi)} p_N(\xi)\, d\xi = \int \overline{\hat{g}(\xi)} \prod_{k=1}^{N} m_o(\xi/2^k)\, d\xi.$$

where g was “nice” and also zero in some interval $[M_1 - \epsilon,\, M_2 + \epsilon]$ where $\epsilon > 0$ depended on g. The product in the integral is a trig polynomial, and its frequencies were found to be in the interval $[M_1 - \epsilon,\, M_2 + \epsilon]$ for all N large enough. Fourier inversion implied the integral was zero. This and a density argument showed that Φ has compact support. Now that Φ has compact support, the fact that it’s in L^2 and the Schwarz Inequality showed that Φ is also in L^1. But then the Fourier transform of Φ is really given by an integral! Hence the function $\widehat{\Phi}(\xi)$ that was once a continuous limit, and then turned out to be in $L^2 \cap L^1$, can be written as

(Phi Facts 2, 3 & 4)
$$\prod_{k=1}^{\infty} m_o(\xi/2^k) = \widehat{\Phi}(\xi) = \int \Phi(t) e^{-it\xi}\, dt$$
$$\text{so}\quad \widehat{\Phi}(0) = \int \Phi(t)\, dt = 1$$
$$\text{and}\quad \widehat{\Phi}(\xi) = m_o(\xi/2)\widehat{\Phi}(\xi/2).$$

Fifth question: Is there a constant K such that $K\Phi$ has norm 1 and has o.n. integer translates?

Here, the answer is "not necessarily, but…" We used a new assumption of Type III that is *sufficient:*

We supposed that

$$(h5) \qquad |\xi| \le \pi/2 \Rightarrow m_o(\xi) \neq 0.$$

We showed that $\{\Phi(t-n) : n \in \mathbb{Z}\}$ is an *orthogonal* set and that orthonormality can then be guaranteed by normalizing.

Sixth question: What is the normalizing factor?

With a bit more work we found that no normalization was needed, and we renamed Φ to φ.

A candidate for a scaling function: $\varphi(t) := \Phi(t)$.

Seventh question: Does this φ determine a MRA? Reconstituted from Hernandez and Weiss, *A First Course on Wavelets.*

The answer is "Yes." We began by defining V_0 to be the closure of the span of the integer translates of φ.

This showed that **MRA (v) holds.**

Next, we defined, for each integer j,

$$V_j := \{f(2^j t) : f(t) \in V_0\}.$$

This showed that **MRA (iv) holds.**

Since V_0 is closed, and each V_j is a dilate of V_0, **MRA (i) holds.**

The formula

$$\widehat{\Phi}(\xi) = m_o(\xi/2)\widehat{\Phi}(\xi/2)$$

was used to show that the spaces V_j are nested, so **MRA (ii) holds.**

To show that (iii) is true there were two things to do. We began by showing that (i), (ii), (iv) and (v) imply that

$$\bigcap_j V_j = \{0\}.$$

In other words, if we know that (i), (ii), (iv) and (v) hold we need not check that the intersection of all the V_j is the zero subspace.

The other half of (iii) is about the density of the subspace $\operatorname{span}(W)$, where $W := \bigcup_j V_j$. We used: φ is integrable, so $\widehat{\varphi}$ is continuous, and we used Phi Fact 3, that $\widehat{\varphi}(0) = 1$.

If $g \perp W$, then for every $f \in W$ and every dyadic rational r we showed that

$$0 = \langle f(t+r),\, g\rangle_{dt} = \int f(t+r)\overline{g(t)}\,dt =: h(r).$$

Then we showed

$$h(x) = \int f(t+x)\overline{g(t)}\,dt$$

is continuous. Since every real number x is the limit of a sequence of dyadic rationals r, this showed $h(x) \equiv 0$. Once more we used the formula $\langle f,\, g\rangle = \frac{1}{2\pi}\langle \hat{f},\, \hat{g}\rangle$, which gave

$$0 = \int f(t+x)\overline{g(t)}\,dt = \frac{1}{2\pi}\int e^{ix\xi}\hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi.$$

The function $\hat{f}(\xi)\overline{\hat{g}(\xi)}$ is integrable and has Fourier transform identically zero. It is a Fourier Fact that this means $\hat{f}(\xi)\overline{\hat{g}(\xi)} = 0$ a.e.

We chose $f(t) = 2^j\varphi(2^jt)$. Then $f \in W$, and $\hat{f}(\xi) = \widehat{\varphi}(2^{-j}\xi)$, so

$$\widehat{\varphi}(2^{-j}\xi)\overline{\widehat{g}(\xi)} = 0 \text{ a.e.}$$

It followed that $\overline{\widehat{g}(\xi)} = 0$ a.e. in $|\xi| < 2^j\delta$. As we let $j \to \infty$, we concluded that $\hat{g}(\xi) = 0$ a.e. Hence, by Fourier Inversion, $g(t) = 0$ a.e. also. This showed that

$$\overline{\bigcup_j V_j} = L^2(\mathbb{R}),$$

and hence that **MRA (iii) holds.** This completed the proof that φ determines a MRA.

Appendix unfinished!

A1: Since $m_o(\xi) = \sum_n \frac{h_n}{\sqrt{2}}e^{-in\xi}$ and only finitely many of the h_n are non-zero, we can differentiate term-by-term:

$$m_o'(\xi) = \sum_n \frac{h_n}{\sqrt{2}}(-in)e^{-in\xi}, \quad \text{so} \quad |m_o'(\xi)| \leq \sum_n \frac{|h_n|}{\sqrt{2}}|n| \quad \text{for all} \quad \xi.$$

Thus we can use this number: $B := \sum_n |n||h_n|/\sqrt{2} < \infty$ because only finitely many h_n are non-zero.

A2: About infinitely differentiable functions with compact support

We begin with the function

$$h(x) := \begin{cases} 0 & \text{if } x \leq 0; \\ e^{-1/x} & \text{if } x > 0. \end{cases}$$

This function is infinitely differentiable if $x \neq 0$. To show that h is infinitely differentiable if $x = 0$ we start by showing that h is continuous at zero: this amounts to showing that $e^{-1/x} \to 0$ as $x \downarrow 0$, which is true since, if we let $u = 1/x$, $x \downarrow 0 \iff u \uparrow +\infty$, and then $e^{-1/x} = 1/e^u \to 0$.

We will use this lemma: *If* $f(x)$ *is defined and continuous for* $x \geq 0$ *(so that* f *is continuous from the right at* 0*), and if* f *is differentiable for* $x > 0$ *and* $f'(x) \to L$ *as* $x \downarrow 0$*, then* f *is differentiable from the right at* 0 *and* $f'(0) = L$*, where* $f'(0)$ *is the derivative from the right, namely* $\displaystyle\lim_{x\downarrow 0}\frac{f(x) - f(0)}{x}$. Proof: later.

For our function $h(x)$, when $x > 0$ we have $h'(x) = e^{-1/x}(1/x^2) = h(x)P_1(1/x)$, where P_1 is the polynomial $P_1(u) = u^2$. Then $h'(x) = h'(1/u) = e^{-u}P_1(u) = P_1(u)/e^u \to 0$ as $u \to \infty$, meaning that $x \to 0$ from the right. By the lemma, this means that h is differentiable from the right at 0, with right-hand derivative 0. But h is also differentiable from the left at 0, with left-hand derivative 0. Thus $h'(0)$ exists and $h'(0) = 0$.

We continue using induction. Suppose we knew that $h^{(n)}(x) = e^{-1/x}P_n(1/x) = h(x)P_n(u)$, where $P_n(u)$ is a polynomial of degree $2n$ in u. Then

$$h^{(n+1)}(x) = \left[h^{(n)}\right]'(x) = e^{-1/x}[(1/x^2)P_n(1/x) - P_n'(1/x)(1/x^2)] = h(x)(1/x^2)[P_n(1/x) - P_n'(1/x)] = h(x)P_{n+1}(1/x),$$

where $P_{n+1}(u) = u^2[P_n(u) - P_n'(u)]$. Since $P_n'(u)$ has degree one less than that of $P_n(u)$, our claim follows. If, therefore, we also know that $h^{(n)}(x) \to 0$ as $x \downarrow 0$, and that $h^{(n)}(0) = 0$ we can show the same is true for $h^{(n+1)}(x)$ by using our claim and l'Hospital's Rule on $h^{(n+1)}(x) = h^{(n+1)}(1/u) = e^{-u}P_{n+1}(u) = P_{n+1}(u)/e^u \to 0$ because e^u tends to infinity faster than *any* polynomial. But as before this means (using the lemma) that $h^{(n+1)}(0) = 0$. Because we know these wo things when $n = 1$, induction shows that they are true for every n, so $h(x)$ is infinitely differentiable, and $h(x) = 0$ if $x \leq 0$, while $h(x) > 0$ if $x > 0$.

We can now construct a function $g(x)$ that is infinitely differentiable, positive if $|x| < 1$ and identically zero if $|x| \geq 1$: we set $g(x) := h(2(1+x))h(2(1-x)) = e^{1/(1-x^2)}$ if $|x| < 1$, while $g(x) \equiv 0$ if $|x| \geq 1$. You should check!

A3: Why $G_n = 0$ if $|n| \geq M_2 - M_1 \ldots$ Although we defined G_n as a Fourier coefficient, it's also true that

$$G_n = \langle\Phi(t-n), \Phi(t)\rangle_{dt} = \int \Phi(t-n)\overline{\Phi(t)}\,dt,$$

and we know that $\Phi(t) = 0$ a.e. if $t \notin [M_1, M_2]$. In order to have $\Phi(t-n)\overline{\Phi(t)} \neq 0$, we need $M_1 \leq t - n \leq M_2$ and $M_1 \leq t \leq M_2$ to be true simultaneously. The first inequality says $t - M_2 \leq n \leq t - M_1$ and from the first inequality we see that $M_1 - M_2 \leq t - M_2 \leq n \leq t - M_1 \leq M_2 - M_1$ so $|n| \leq M_2 - M_1$ if $G_n \neq 0$.