§1 Lebesgue theory - an overview: Introduction

The theory of the Lebesgue integral is more complicated to learn than the theory of the Riemann integral. But the Lebesgue integral has distinct advantages, both theoretical and "practical." First, a function $f(t)$ is Riemann integrable on a bounded closed interval $[a, b]$ if and only if $f(t)$ is bounded and continuous almost everywhere (a.e.). Items (31) and (32) in the list below explain "a.e.," though briefly. For now, it means that $f(t)$ must be continuous except in a set of zero "length." Piecewise continuous functions, and functions that are continuous except at a sequence $\{t_n\}$ of points are continuous a.e. In contrast to these requirements, a Lebesgue integrable function does not have to be bounded and does not have to be continuous *anywhere*. This crucial difference carries over into limit situations. The main general requirement that a sequence of Riemann integrable functions have a Riemann integrable limit is that the convergence be uniform. This is a very stringent requirement. The requirements for sequences of Lebesgue integrable functions are not as stringent, are harder to justify, but are *much easier to check in practice!*

A list of facts: Lebesgue theory

There are three attributes of important facts in the Lebesgue theory – "fundamental," "basic" and "useful." Fundamental Facts are essential for understanding the theory, but they don't add much for the confident user of Useful Facts. Basic Facts are the ones an experienced person would expect to be true. Some Facts have more than one of these attributes.

This note is therefore organized in three parts: first, Useful Facts, then Basic Facts, then Fundamental Facts. You may need to look in the last list to find the definition of some term used in the first list.

§2 Useful Facts

Foreword: An integral is finite and well-defined if and only if the same integral, with absolute values on the integrand, is finite. This is really a Fundamantal Definition that it is essential to know about!

Note on notation: An integral is always taken over a measurable set, for example, $\int_E f(x)\,dx$. But, unless we need to emphasize this, we'll just write $\int f(x)\,dx$. Just about any set you'd normally consider is measurable. It's actually very difficult to "construct" non-measurable sets!

Why we need convergence theorems

When we use the Riemann integral the interchange of limit and integral is usually done only when convergence is uniform. The Lebesbue theory offers much more flexibility. The price we pay for the flexibility is having to know more convergence theorems.

After a cautionary note, we list some convergence theorems as Useful Facts.

Given a sequence $\{f_n\}$ of measurable functions, that converges a.e. (see "Almost everywhere," near the end of Fundamental Facts) to a limit function $f(x)$, it may, or may not, be true that the integral of the limit has the same value as the limit of the integrals, even if the latter limit exists. For example, define $f_n(x) := 1/n$ for $0 \le x \le n$, and define $f_n(x) := 0$ otherwise. Then the f_n are measurable, and they converge to zero, but their integrals converge to 1. Another example: define $f_n(x) := (-1)^n/n$ for $0 \le x \le n$, and define $f_n(x) := 0$ otherwise. Then the f_n are measurable, and they converge to zero, but the sequence of their integrals does not converge to a limit.

(1) **Dominated Convergence Theorem (Lebesgue)**
Given: a sequence $\{f_n\}$ of measurable complex-valued functions, that converges a.e. to a limit function $f(x)$.
If there exists a measurable function $g(x) \ge 0$ a.e., such that

$$\text{(i)} \quad \int g(x)\,dx < \infty,$$

and

$$\text{(ii) for each } n, \quad |f_n(x)| \le g(x) \text{ a.e.,}$$

then

$$\lim_{n\to\infty} \int f_n(x)\,dx = \int \lim_{n\to\infty} f_n(x)\,dx = \int f(x)\,dx,$$

and $\displaystyle\int |f(x)|\,dx < \infty$. This is Lebesgue's Dominated Convergence Theorem. The function g is called the "dominating function." This theorem includes the series version, because a series can be viewed as the integral of a function that is constant on consecutive intervals of unit length.

(2) **Monotone Convergence Theorem (Lebesgue)**
Given: a sequence $\{f_n\}$ of measurable real-valued functions, that is a.e. monotone increasing, i.e., $f_n(x) \le f_{n+1}(x)$, increases a.e. to a limit function $f(x) \le +\infty$.
If there exists a measurable function $g(x)$ such that
(i) $\int |g(x)|\,dx < \infty$,
and
(ii) for each n, $f_n(x) \ge g(x)$ a.e.,
then

$$\lim_{n\to\infty} \int f_n(x)\,dx = \int \lim_{n\to\infty} f_n(x)\,dx = \int f(x)\,dx,$$

where $f(x) := \lim_{n\to\infty} f_n(x)$. This is Lebesgue's Monotone Convergence Theorem. In usual applications, $g(x) \equiv 0$. Again, this theorem includes its series version.

Fatou's Lemmas: Sometimes we know more about the *integrals* of a sequence of functions than we do about the functions themselves. Fatou's Lemma is then available. Fatou's Lemma has an intimidating thing called the "lower limit," of the sequence denoted $\liminf_{n\to\infty} f_n(x)$. It is what you get if you take the lower envelope of all the graphs in the sequence but use only functions with large subscripts, then let "large" tend to infinity. Fatou's Lemma says that integral of the "liminf" of the sequence exists and satisfies an inequality. There is an easier version useful if all we know is that the limit exists.

(3) **Fatou's Lemma** (full version)

Suppose that $\{f_n(x)\}$ is a sequence of measurable functions defined a.e. on a measurable set E, and that there is some function $g(x)$, *integrable* on E, such that $f_n(x) \ge g(x)$ a.e., then

$$\int_E \liminf_{n\to\infty} f_n(x)\,dx \le \liminf_{n\to\infty} \int_E f_n(x)\,dx.$$

(3′) **Fatou's Lemma** ('lite' version)

Suppose that $\{f_n(x)\}$ is a sequence of measurable functions defined a.e. on a measurable set E, and that there is some function $g(x)$, *integrable* on E, such that $f_n(x) \ge g(x)$ a.e., then

$$\int_E \lim_{n\to\infty} f_n(x)\,dx \le \lim_{n\to\infty} \int_E f_n(x)\,dx,$$

in case the indicated limits exist.

The preceding Theorems had to do with interchanging two limit operations: pointwise limit and integral (which is a limit operation). Next we consider interchange of integrals.

Some theorems on multiple integrals

(4) **Fubini's Theorem** Double integrals and iterated integrals are equal if any one of them is finite when the same integral, with absolute values on the integrand, is finite. In terms of formulas,

$$\iint f(x,y)\,dA = \int \left(\int f(x,y)\,dx \right) dy = \int \left(\int f(x,y)\,dy \right) dx$$

if any one of

$$\iint |f(x,y)|\,dA, \quad \int \left(\int |f(x,y)|\,dx \right) dy, \quad \int \left(\int |f(x,y)|\,dy \right) dx$$

is finite: this is Fubini's Theorem. This is also true when there are more integrals, and when the integrals are replaced by multiple sums. There are examples of functions that do not have the same iterated integrals when we change the order of integration: $f(x,\, y) := 2(x-y)/|x-y|$, if $0 < |x-y| < 1$, and $f(x,\, y) := 0$ otherwise.

If E is a measurable set, $f(x)$ is a measurable function defined a.e. on E, and $\int_E f(x)\,dx$ exists, then, if $\{E_n\}$ is a sequence of measurable subsets that are pairwise disjoint, and such that $E = \bigcup_{n=1}^{\infty} E_n$, then

$$\int_E f(x)\,dx = \sum_{n=1}^{\infty} \int_{E_n} f(x)\,dx.$$

(4′) **Tonelli's Theorem** Double integrals and iterated integrals are equal if the integrand is a non-negative measurable function.

The next Theorem is not really about interchanging orders of integration, but an interchange occurs in it. Instead, it turns a sum into an integral, and involves "L^p" spaces...

(5) **Minkowski's integral inequality**

This is a "continuous" version of the triangle inequality; it says that "the norm of a sum is less than or equal to the sum of the norms," where the "sum" is an integral. In the version we will use, the "norms" referred to are integrals of the form

$$\|f\|_p = \left\{ \int |f(x)|^p dx \right\}^{1/p},$$

where f is measurable and $1 \leq p < \infty$. The symbol $\|f\|_p$ is read "the L-p norm of f." Minkowski's integral inequality is

$$\left\{ \int_A \left| \int_B F(x,\, y)\, dy \right|^p dx \right\}^{1/p} \leq \int_B \left\{ \int_A |F(x,\, y)|^p \; dx \right\}^{1/p} dy,$$

where A and B are measurable sets, $F(x,\, y)$ is a measurable function on the Cartesian product of A and B, namely the set $A \times B$ of *all* ordered pairs $(x,\, y)$ with $x \in A$ and $y \in B$. We can replace either (or both) of the integrals by a series:

$$\left\{ \int_A \left| \sum_n F_n(x) \right|^p dx \right\}^{1/p} \leq \sum_n \left\{ \int_A |F_n(x)|^p \; dx \right\}^{1/p} dy$$

is one possibility.

(6) **Differentiation of integrals**

If $f(x)$ is measurable, and $\displaystyle\int_a^b |f(x)|\,dx < \infty$ for all bounded intervals $(a,\, b)$, then for a.e. x_o,

$$f(x_o) = \lim_{\substack{0 < b-a \to 0 \\ a \leq x_o \leq b}} \frac{1}{b-a} \int_a^b f(x)\,dx. \tag{7}$$

This is Lebesgue's Differentiation Theorem. Lebesgue's Differentiation Theorem is true in more than one variable, but the intervals have to be replaced by cubes or balls, or other sets with a somewhat regular shape; in particular, rectangles of arbitrary shape that contain x_o, and shrink down to x_o, need not work!

Differentiation under the integral sign

Given that $f(x) := \int_E F(x,y)\,dy$, where

$\int_E |F(x,y)|\,dy < \infty$ for all $x \in (a,\, b)$, and $\displaystyle\int_a^b \int_E \left| \frac{\partial F(x,y)}{\partial x} \right| dy\,dx < \infty$, it is true that

$$f'(x) = \int_E \frac{\partial F(x,y)}{\partial x}\,dy \text{ almost everywhere in } (a,\, b), \tag{8}$$

and the derivative is sure to exist at a given point x_o if the difference quotient $\frac{F(x_o+h,y)-F(x,y)}{h}$ converges a.e. in E and is bounded, in absolute value, by a function $g(y)$ that is non-negative and integrable on E.

(9) **Absolute continuity**
In particular, if $f(x) := \int_a^x g(t)\,dt$, and g is integrable over every bounded interval, or just over some bounded open interval that contains $[a,\, b]$, then $f'(x) = g(x)$ a.e. in the relevant interval, and the equation holds for sure at x_o if g is continuous at x_o. This is as close as we can get to the Fundamental Theorem of Calculus in the Lebesgue theory. There is an epsilon-delta way to express absolute continuity, which will not concern us here.

(10) **Continuity of norms with respect to translation**
If $\left\{\int |f(x)|^p\,dx\right\}^{1/p} < \infty$, where $1 \le p < \infty$, then $\lim_{h\to 0} \left\{\int |f(x+h) - f(x)|^p\,dx\right\}^{1/p} = 0$. That is, *translation is continuous* on $L^p(\mathbb{R})$. This is also true in several variables.

However, translation need not be continuous on $L^\infty(\mathbb{R}^n)$, for any n. The meaning of $L^\infty(\mathbb{R}^n)$: measurable functions $f(x)$ that are essentially bounded, meaning that there is a number B such that the set of all x where $|f(x)| > B$ is a set of measure zero. The smallest of all such B is called the essential upper bound for $|f(x)|$, and it is often denoted $\|f\|_\infty$.

§3 **Basic Facts**

(11) The sum, difference, product and scalar multiples of measurable real or complex-valued functions $f(x)$ and $g(x)$ are measurable functions.

(12) The maximum and minimum of measurable real-valued functions $f(x)$, $g(x)$ are measurable.

(13) If a sequence $f_n(x)$ of measurable functions converges a.e. to a limit function $f(x)$, then $f(x)$ is a measurable function.

Continuous functions are measurable, so every function that can be represented as the a.e. limit of a sequence of continuous functions is measurable. In particular, some functions that are not Riemann integrable can be represented as the a.e. limit of a sequence of continuous functions.

(14) **The "existence" of integrals – a tricky definition:** Lebesgue integrals are *always* defined on the basis of integrals of non-negative functions. If a measurable function is real-valued, and is not of constant sign, we split the function into its "positive" and "negative" parts first, and find the integral of each part. We then say "the integral exists" if at least one of those integrals is finite, and the value of the integral is the integral of the positive part minus the integral of the negative part. In case the integrals of both parts are infinite, we say "the integral does not exist." Thus the integral of a non-negative measurable function always "exists," but the integral may be infinite in value. For example, the integral $\int_0^\infty 1\,dx = +\infty$. The integral of a non-negative measurable function is the (Lebesgue) area of the set that lies under its graph and above the x-axis. If a measurable function is complex-valued, we calculate its intergral by working with the real and imaginary parts separately, then reassemble to get the complex value of the integral as a sum now of four terms, with i's used at appropriate places.

(15) **Integrable functions – another tricky definition:**

We say that a measurable function $f(x)$ is *integrable* if (1) its integral exists, and (2) its integral is finite. In the case of complex-valued functions, this means that *four* integrals have to be finite: the integrals of the positive and negative parts of the real part of the function, and the integrals of the positive and negative parts of the imaginary part of the function. This is not as hard as it sounds, because a complex-valued function is integrable if and only if the integral of its absolute value is finite. In other words, a function $f(x)$ is integrable if and only if $|f(x)|$ is integrable.

Integrable functions – the usual facts:

(16) The sum, difference, maximum and minimum (in the real-valued case), and scalar multiples of integrable real or complex-valued functions $f(x)$, $g(x)$ are integrable, and

$$\int (f(x) \pm g(x))dx = \int f(x)\,dx \pm \int g(x)\,dx, \quad \int cf(x)\,dx = c\int f(x)\,dx.$$

The product, $f(x)g(x)$, of integrable real or complex-valued functions $f(x)$, $g(x)$ need not be integrable!

The pointwise limit of a sequence of integrable functions, if it exists, need not be integrable, though it *is* measurable.

However, we have Fatou's Lemma (see (3) and (3′)), which gives a useful criterion for integrability in this situation. Here is an example that shows why we need Fatou's Lemma.
Example: Define $f_n(x) := 1$ for $|x| < n$, and $f_n(x) := 0$ for $|x| \geq n$. Then $f_n(x) \to 1$ for all x, and $\int_{\mathbb{R}} 1\, dx$ is infinite, so the limit function is not integrable (its integral does exist, though).

(17) If $f(x)$ is integrable on a measurable set E, then $\left|\int_E f(x)\, dx\right| \leq \int_E |f(x)|\, dx$.

(18) If $f(x) \geq 0$ and $f(x)$ is measurable, and $\int f(x)\, dx = 0$, then $f(x) = 0$ a.e.

(19) If the integrals of $f(x)$ and $g(x)$ both exist, and if $f(x) \leq g(x)$ a.e., then $\int f(x)\, dx \leq \int g(x)\, dx$.

§4 Fundamental Facts and Definitions

(20) A set G of real numbers is *open* if, whenever $x_o \in G$, there exists $\delta > 0$ such that $(x_o - \delta, x_o + \delta) \subseteq G$. That is, *all* the points that are sufficiently close to x_o are also in G.

(21) A set F of real numbers is *closed* if, whenever a sequence x_n of points in F has a limit x_ω, then $x_\omega \in F$. That is, limits of sequences in F are still in F.

(22) A set is open if and only if its complement is closed; a set is closed if and only if its complement is open. The set $(0, 1)$ is open, but not closed; the set $[0, 1]$ is closed, but not open. The set $[0, 1)$ is neither open nor closed.

(23) A set E is *covered* by a family $\mathcal{C}$ of sets (a *family* of sets is a set of sets; "family" is a synonym for "set") if, for every $x_o \in E$, there is a set $C \in \mathcal{C}$ such that $x_o \in C$. For example, given any $\epsilon > 0$, $\mathbb{Z}$ is covered by the family $\mathcal{C}_\epsilon$ that consists of the open intervals $C_m := (m - \epsilon 2^{-|m|-3},\, m + \epsilon 2^{-|m|-3})$. Note that the sum of the lengths of the intervals in $\mathcal{C}_\epsilon$ is $\frac{\epsilon}{4} + 2\epsilon \sum_{m=1}^{\infty} 2^{-m-2} < \epsilon$.

(24) A subset E of $\mathbb{R}$ is *measurable* if, for every $\epsilon > 0$, there exist an open set G_ϵ and a closed set F_ϵ such that $F_\epsilon \subseteq E \subseteq G_\epsilon$, and such that the open set $G_\epsilon \setminus F_\epsilon$ can be covered by open intervals (a_n, b_n), $n \in \mathbb{Z}$, in such a way that $\sum_{n \in \mathbb{Z}}^{\infty} (b_n - a_n) < \epsilon$. Intervals, both bounded and unbounded, are measurable, the empty set and $\mathbb{R}$ are measurable, the union of, and the intersection of, a *sequence* of measurable sets is measurable. Open sets, and closed sets, are measurable.

The collection of all measurable sets is called a *sigma algebra* of sets.
(25) **Definition** A family $\mathcal{S}$ of subsets E of a set X is a sigma algebra if
(a) $X \in \mathcal{S}$,
(b) If $E \in \mathcal{S}$, then its complement, E^c, also belongs to $\mathcal{S}$,
(c) If $E_n \in \mathcal{S}$, for $n = 1, 2, 3, \ldots$ then $\bigcup_{n=1}^{\infty} E_n \in \mathcal{S}$.

This is the whole definition! But it does not tell the whole story, so lets continue the definition by adding some conditions that actually follow from the conditions (a) (b) (c).
(d) The empty set is in $\mathcal{S}$,
(e) If $E_n \in \mathcal{S}$, for $n = 1, 2, 3, \ldots$ then $\bigcap_{n=1}^{\infty} E_n \in \mathcal{S}$.
(f) If $E_1 \in \mathcal{S}$ and $E_2 \in \mathcal{S}$ then $E_1 \setminus E_2 \in \mathcal{S}$, where $E_1 \setminus E_2$ denotes the set of all points that are in E_1 but not in E_2.

Measurable subsets of $\mathbb{R}^n$ are defined in a way that is similar to the way they are defined in $\mathbb{R}$, but the definitions of open set and closed set have to reflect the different notions of distance that are used there. We usually use Euclidean distance: the distance from $(x_1, \ldots, x_n)$ to $(y_1, \ldots, y_n)$ is $\sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$, and we often write

$$|x - y| \text{ or } \|x - y\| \text{ for } \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

Instead of intervals, we use "boxes," "cubes," and " balls" to define various things in $\mathbb{R}^n$. A "box" has its edges parallel to the coordinate axes, and the edges can have any finite lengths, including zero. A "cube" is a box whose edges all have the same length. We most often work with open or closed cubes; closed cubes contain all the points on the sides and edges, open cubes contain none of them. A ball can be open or closed; most often we work with open balls. The "open ball" of radius δ and center x_o, denoted $B_\delta(x_o)$, is the set of all $x \in \mathbb{R}^n$ such that $\|x - x_o\| < \delta$.

That is, the set of all points in $\mathbb{R}^n$ that are within a distince δ of x_o. This is the analog, in $\mathbb{R}^n$, of the interval $(x_o - \delta, x_o + \delta)$ in $\mathbb{R}$.

Then a set $G \subseteq \mathbb{R}^n$ is open if, whenever $x_o \in G$, there exists $\delta > 0$ such that $B_\delta(x_o) \subseteq G$. That is, *all* the points that are sufficiently close to x_o are also in G. The *measure* of a measurable set E is denoted in too many ways – $\lambda(E)$ and $|E|$ are commonly used. The measure of the empty set is 0: $\lambda(\emptyset) = 0$. For intervals, the measure is the length: $\lambda((a,b)) = b - a = \lambda([a,b)) = \lambda((a,b]) = \lambda([a,b])$ if the interval is bounded. The measure of an unbounded interval is infinite: $\lambda([0,\infty)) = +\infty$, or in the other notation, $\big|[0,\infty)\big| = +\infty$.

For more complicated sets, there are rules that help: If measurable sets E_n, $n \in \mathbb{Z}$, are disjoint unless their subscripts are equal, then the measure of their union is the sum of their measures:

(26) $$\text{If } E_n \cap E_m = \emptyset \text{ whenever } n \neq m, \text{ then } \Big|\bigcup_{n\in\mathbb{Z}} E_n\Big| = \sum_{n\in\mathbb{Z}} |E_n|.$$

Since we can let all but 2 of the sets E_n be empty, this shows that,

(27) $$\text{if } E_1 \cap E_2 = \emptyset, \text{ then } |E_1 \cup E_2| = |E_1| + |E_2|,$$

provided that the two sets in question are measurable.

In what follows, let us assume all the sets mentioned are measurable!

The Facts underlying the properties of an integral as the measure of the set beneath a graph

If sets increase monotonically, their measures behave as we would expect:

(28) $$\text{if } E_n \subseteq E_m \text{ whenever } n \leq m, \text{ then } \left|\bigcup_{n\in\mathbb{Z}} E_n\right| = \lim_{n\to\infty} |E_n|.$$

If sets *decrease* monotonically, their measures may *not* behave as we would expect! However,

(29) $$\text{if } E_n \supseteq E_m \text{ whenever } n \leq m, \text{ and } |E_N| < \infty \text{ for some } N, \text{ then } \left|\bigcap_{n\in\mathbb{Z}} E_n\right| = \lim_{n\to\infty} |E_n|.$$

A cautionary example: if $E_N = (N, +\infty)$ then all the sets have infinite measure and (29) is no longer true.

(30) If $E_1 \subseteq E_2$, and at least one of them has finite measure, then $|E_2 \setminus E_1| = |E_2| - |E_1|$.

(31) **"Almost everywhere," and "sets of measure zero," or "null sets"**

A statement, such as $f(x) = g(x)$, is true *almost everywhere*, or *a.e.*, or *for almost all* x, if the set of all x for which the statement is false, i.e., $f(x) \neq g(x)$, is a set of Lebesgue measure zero.

(32) We do not need Lebesgue measure to define "set of Lebesgue measure zero!" A set Z has Lebesgue measure zero if and only if for all $\epsilon > 0$, Z can be covered by a family of open intervals (a_k, b_k), $k \in \mathbb{Z}$, such that $\sum_{k\in\mathbb{Z}}(b_k - a_k) < \epsilon$. It is important to notice that the family of intervals used here depends on ϵ! To define sets of measure zero, or *null sets,* in $\mathbb{R}^n$ we use cubes Q_k instead of intervals and require that $\sum_{k\in\mathbb{Z}} \lambda_k^n < \epsilon$, where λ_k is the edge-length of Q_k, so $\lambda_k^n = |Q_k|$.

The example of a covering of $\mathbb{Z}$ by the intervals $(k - \epsilon 2^{-|k|-3}, k + \epsilon 2^{-|k|-3})$ is a model for showing that every countable set is a set of measure zero. But there are uncountable sets of measure zero; the Cantor "middle thirds" set is an example.

An example of what we can use this jargon for is:

(33) An integrable function is finite a.e.: $\int |f(t)|\, dt < \infty \Rightarrow |f(t)| < \infty$ almost everywhere.

Measurable functions

(34) A function $f : \mathbb{R} \to \mathbb{R}$ is a *measurable function* if, for all $y \in \mathbb{R}$, the set $\{x \in \mathbb{R} : f(x) > y\}$ is a measurable set.

The indicator function of a set E is a measurable function if and only if the set E is measurable. Linear combinations of measurable functions are measurable functions, as well as pointwise a.e. limits of sequences of measurable functions.

(35) A function $f : \mathbb{R} \to \mathbb{C}$ is a *measurable function* if the real and imaginary parts of f are each measurable functions.

(36) Continuous functions, and functions that are continuous a.e. are measurable.

The "existence" of Lebesgue integrals

(37) The *Lebesgue integral* of a *non-negative* measurable function $f : \mathbb{R} \to \mathbb{R}$ is the Lebesgue measure of the set of points (x, y) that are above or on the x-axis, and below the graph of f. This set might have infinite measure! We define $\int f(x)\, dx$ to be the Lebesgue measure of the "area under the graph of f."

(38) If a function $f(x)$ is continuous a.e. and bounded, and $f(x) = 0$ outside some bounded interval $[a, b]$, then its Lebesgue integral, $\int f(x)\, dx$ over all of $\mathbb{R}$ is the same as its Riemann integral $\int_a^b f(x)\, dx$.

(39) We can express every measurable real-valued function as $f(x) = f(x)^+ - f(x)^-$, where, for every real number y, $y^+ := \max(0, y)$, $y^- := \max(0, -y)$, so that $y = y^+ - y^-$, and $|y| = y^+ + y^-$. We say that *the integral a measurable real-valued function exists* if at least one of the integrals $\int f^+(x)\, dx$ and $\int f^-(x)\, dx$ is finite, and then we write $\int f(x)\, dx := \int f^+(x)\, dx - \int f^-(x)\, dx$.