From Rudin (please read this part with the book handy):
3.7 Theorem *The subsequential limits of a sequence* $\{p_n\}$ *in a metric space* X *form a closed subset of* X.

3.16 Definition (paraphrased): *Let* $\{s_n\}$ *be a sequence of real numbers. Let* E *be the set of subsequential limits (in the extended-real number system) of* $\{s_n\}$. *Thus* E *may contain* $+\infty$ *or* $-\infty$, *and now* $\sup E$ *and* $\inf E$ *both exist. We define*

$$\limsup_{n\to\infty} s_n := \sup E \quad \textit{and} \quad \liminf_{n\to\infty} s_n := \inf E.$$

Next we look at the version of Theorem 3.17 that was given in class. Let's use the notation $\overline{\mathbb{R}}$ to stand for the extended-real "number" system.

Theorem *Let* $\{s_n\}$ *be a sequence of real numbers. Then* $\alpha \in \overline{\mathbb{R}}$ *is equal to* $\limsup_{n\to\infty} s_n$ *if and only if*

(1) *for all* $\beta > \alpha$ *there exists* $N \in \mathbb{N}$ *such that for all* $n \geq N$, $s_n < \beta$

and

(2) *for all* $\beta < \alpha$ *and for all* $N \in \mathbb{N}$ *there exists* $n > N$ *such that* $s_n > \beta$.

To prove this Theorem we recall that $\overline{\mathbb{R}}$ is a compact metric space (because its topology is imported from $[0,\,1]$, viewed as a subset of $\mathbb{R}$) with respect to the metric

$$d(x,\,y) := \begin{cases} \left|\frac{x}{1+|x|} - \frac{y}{1+|y|}\right|, & \text{if } x \in \mathbb{R} \text{ and } y \in \mathbb{R}; \\ \left|\frac{x}{1+|x|} - 1\right|, & \text{if } x \in \mathbb{R} \text{ and } y = +\infty; \\ \left|1 - \frac{y}{1+|y|}\right|, & \text{if } x = +\infty \text{ and } y \in \mathbb{R}; \\ \left|\frac{x}{1+|x|} + 1\right|, & \text{if } x \in \mathbb{R} \text{ and } y = -\infty; \\ \left|1 + \frac{y}{1+|y|}\right|, & \text{if } x = -\infty \text{ and } y \in \mathbb{R}; \\ 0, & \text{if } x \text{ and } y \text{ are the same infinity,} \\ 2, & \text{if } x \text{ and } y \text{ are opposite infinities.} \end{cases}$$

Then by Theorem 3.6, $\{s_n\}$ has a convergent subsequence in $\overline{\mathbb{R}}$, so E is not empty. By Theorem 3.7, E is closed, so $\limsup_{n\to\infty} s_n \in E$. By definition of E, there exists a subsequence $\{s_{n_k}\}_{k=1}^{\infty}$ such that $s_{n_k} \to \limsup_{n\to\infty} s_n$. Let's set $\alpha = \limsup_{n\to\infty} s_n = \lim_{n\to\infty} s_{n_k}$.

We can prove (2) first. Suppose $\beta < \alpha$. Let $\epsilon := \alpha - \beta$. Then there exists $K \in \mathbb{N}$ such that for all $n \geq K$, $|s_{n_k} - \alpha| < \epsilon$, so that $s_{n_k} - \alpha > -\epsilon = \beta - \alpha$ for all $n \geq K$. That is, $s_{n_k} > \beta$ for all $n \geq K$. This proves (2). **Caution!** Notice that it's only a *subsequence* that's bigger than β. There may also be infinitely many n such that $s_n < \beta$!

We can prove (1) by contradiction. First, tho, we notice that since $\beta > \sup E$, we are in the case when $\alpha < +\infty$, and so $\beta \notin E$. To get our contradiction, we will construct a subsequence of $\{s_n\}$ that has a limit that is at least β, which gives an element of E that is larger than $\sup E$.

Thus we suppose that (1) is false, and this means that

$$\text{there exists} \quad \beta > \alpha \quad \text{such that for all} \quad N \in \mathbb{N}, \quad \text{there exists} \quad n > N \quad \text{such that} \quad s_n \geq \beta.$$

We proceed by induction. Starting with $N = 1$, we find $n_1 > 1$ such that $s_{n_1} \geq \beta$. We then put $N = n_1$, and so obtain $n_2 > n_1$ such that $s_{n_2} \geq \beta$. Having found $1 < n_1 < n_2 < \cdots < n_k$ such that $s_{n_j} \geq \beta$ for $1 \leq j \leq k$, we set $N = n_k$ and so obtain $n_{k+1} > n_k$ such that $s_{n_{k+1}} \geq \beta$. This gives us a subsequence $s_{n_k} \geq \beta$. Since a subsequence is a sequence in its own right, s_{n_k} has a convergent-in-$\overline{\mathbb{R}}$ subsequence $s_{n_{k_\ell}}$, whose limit, an element of E, is necessarily at least β. This contradiction essentially completes the proof of the Theorem.

We are now ready for our third view of limsup. The first view was Rudin's original definition. The second view is given by the Theorem just finished. There is a little bit left 4U: show that if α satisfies (1) and (2) then $\alpha = \limsup_{n\to\infty} s_n$. This is what we need next!

Third view of limsup and liminf

Theorem

$$\limsup_{n\to\infty} s_n = \inf_{n\in\mathbb{N}} \sup_{m\geq n} s_m = \lim_{n\to\infty} \sup_{m\geq n} s_m, \quad \text{and,} \quad \liminf_{n\to\infty} s_n = \sup_{n\in\mathbb{N}} \inf_{m\geq n} s_m = \lim_{n\to\infty} \inf_{m\geq n} s_m.$$

We'll only prove this for limsup. We will use the Second View. I recommend, *strongly*, that you write up the proofs of all of this for liminf yourself!

Let's define $A_n := \sup\limits_{m\geq n} s_m$. I call this the "above-sequence," because $A_n \geq s_n$. Since $A_n = \max(s_n, A_{n+1})$ (check it!), $\{A_n\}$ is a decreasing sequence. Therefore $\alpha := \lim\limits_{n\to\infty} A_n$ exists in $\overline{\mathbb{R}}$.

Let us show that α satisfies (1). Suppose $\beta > \alpha$. Then we can let $\epsilon := \beta - \alpha$, and find N such that $n \geq N \Rightarrow |A_n - \alpha| < \epsilon$. But then $A_n - \alpha < \epsilon = \beta - \alpha$, so $A_n < \beta$ for all $n \geq N$. But then $s_n \leq A_n < \beta$ for all $n \geq N$, so (1) holds.

To show that (2) holds for α, suppose $\beta < \alpha$ and let N be given. Here we put $\epsilon := \alpha - \beta$ and find N_o such that for all $n \geq N_o$, $|A_n - \alpha| = A_n - \alpha > -\epsilon$, which implies (4U) $A_n > \beta$ for all $n \geq N_o$. Now choose $n_o = 1 + \max(N, N_o)$. We notice that n_o depends on β.

Set $\epsilon_o := A_{n_o} - \beta$. Since $A_{n_o} - \epsilon_o < A_{n_o} = \sup_{m\geq n_o} s_m$, there exists $m_o \geq n_o$ such that $s_{m_o} > A_{n_o} - \epsilon_o = A_{n_o} - (A_{n_o} - \beta) = \beta$. We will take $n := m_o$ as our $n > N$ such that $s_n > \beta$. This completes the proof of (2).

Thus by the Theorem about the Second View,

$$\limsup_{n\to\infty} s_n = \inf_{n\in\mathbb{N}} \sup_{m\geq n} s_m = \lim_{n\to\infty} \sup_{m\geq n} s_m.$$

Remark The "above-sequence" $A_n = \sup\limits_{m\geq n} s_m$ and the "below-sequence" $B_n := \inf\limits_{m\geq n} s_m$ are respectively decreasing and increasing, and $B_n \leq s_n \leq A_n$ for all n. But even more is true:

$$\text{for each } n, \text{ it is true that for all } \quad m \geq n, \quad B_n \leq s_m \leq A_n.$$

In geometric terms, this says that the whole tail of the sequence $\{s_m\}$, from n on, namely $\{s_m\}_{m=n}^{\infty}$, lies in the interval $[B_n, A_n]$. That is, the tails are trapped in smaller and smaller intervals, that shrink down toward $[\liminf\limits_{n\to\infty} s_n, \limsup\limits_{n\to\infty} s_n]$. There *can* be infinitely many terms of the sequence that lie outside $[\liminf\limits_{n\to\infty} s_n, \limsup\limits_{n\to\infty} s_n]$. If so, the only limit points that the set of *those* terms can have are $\liminf\limits_{n\to\infty} s_n$ and/or $\limsup\limits_{n\to\infty} s_n$.

Remark Maybe this should be a *fourth* view! We can call an element $\alpha \in \overline{\mathbb{R}}$ a *tail upper bound* for a sequence $\{s_n\}$ if there exists $N_o \in \mathbb{N}$ such that for all $n \geq N_o$ we have $s_n \leq \alpha$. We notice that if $\beta > \alpha$ and α is a tail upper bound for $\{s_n\}$ then β is a tail upper bound for $\{s_n\}$. Thus the set Tub of all tail upper bounds for $\{s_n\}$ is non-empty in $\overline{\mathbb{R}}$ because it must contain $+\infty$. In a similar vein Tub has a greatest lower bound *in* $\overline{\mathbb{R}}$. We can then say

$$\limsup_{n\to\infty} s_n = \inf Tub.$$

What does it mean for $\alpha \in \overline{\mathbb{R}}$ *not* to be in Tub? It means that for all $N_o \in \mathbb{N}$ there exists $n \geq N_o$ such that $s_n > \alpha$. Thus β belonging to Tub means (1) holds for β, tho not strictly, and β not belonging to Tub means (2) holds for β. But if $\beta \in Tub$ and $\beta > \inf Tub$ then (1) holds for β, but not (2).