The Logarithm: Exponential in Reverse

The overall aim of this note is to develop the logarithm function $\log y$, also known as $\ln y$, by "inverting" the exponential function $f(x)$ that we developed in "The Exponential Saga." Inverting here means that we undo the effect of the exponential function: $x \mapsto f(x) \mapsto \log f(x) = x$. A lot of what we do here amounts to naming something and then recognizing when some "formula" or expression can be written down in terms of the name.

For positive natural numbers n we defined $f_n(x) := \left(1 + \frac{x}{n}\right)^n$, and we proved that

$$f(x) := \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n \quad \textit{exists for all} \quad x \in \mathbb{R}. \tag{0.1}$$

We proved that for all real numbers u and v, $f(u+v) = f(u)f(v)$, and we proved that $f(x)$ is continuous, positive, strictly increasing, differentiable and that $f'(x) = f(x)$ for all $x \in \mathbb{R}$. We also showed that $f(0) = 1$, and that $f(x)$ tends to 0 at $-\infty$ and tends to $+\infty$ at $+\infty$. We can write e^x for $f(x)$ if we wish.

One thing we did *not* prove was that, for *every* positive real number y there exists a real number x such that $f(x) = y$. However, the way we proved that $f(x)$ tends to 0 at $-\infty$ and tends to $+\infty$ at $+\infty$ was to work with $e^n = f(1)^n = f(n)$, where $n \in \mathbb{Z}$. We showed, in effect, that the two-way infinite sequence $\{f(n)\}$ is strictly increasing from 0 at $-\infty$ to $+\infty$ at $+\infty$. Therefore every positive number y lies in a unique interval of the form $[f(n),\, f(n+1))$.

One goal of this note is to prove that for all $y > 0$ the equation $f(x) = y$ has a unique solution $x \in \mathbb{R}$. We will prove this as an application of a Very Important Theorem about continuous functions on closed bounded intervals:

(1.1) **The Intermediate Value Theorem.** *Suppose that* $F : [a,\, b] \to \mathbb{R}$ *is continuous and that for some real number* y *we have* $F(a) < y < F(b)$. *Then there exists* c, $a < c < b$, *such that* $F(c) = y$.

Proof: We will actually find the *largest* $c \in [a,\, b]$ such that $F(x) \geq y$ for all $x \in [c,\, b]$. We will make strong use of the Completeness Axiom.

We define the set $S := \{x \in [a,\, b] : F(x) < y\}$. This is a non-empty set because $F(a) < y \Rightarrow a \in S$. And the set is trivially bounded above by b.

We will find it handy to know that there is a number $x_1 < b$ such that $(x_1,\, b]$ contains no points of S. Then x_1 is also an upper bound for S. The reason this is handy is because now we know thaere is an upper bound of S with points to the right of it that are less than b.

To prove these things about some x_1 we will use continuity for the first of five times. In this use we choose $\epsilon := F(b) - y$, positive by hypothesis. Then there exists $\delta > 0$ such that (F is given to be continuous)

$$x \in [a,\, b] \;\&\; |x - b| < \delta \Rightarrow |F(x) - F(b)| < \epsilon, \quad \textbf{or:} \quad b - \delta < x \leq b \;\Rightarrow |F(x) - F(b)| < \epsilon.$$

That is,

$$b - \delta < x \leq b \;\Rightarrow F(x) - F(b) > -\epsilon = y - F(b), \quad \text{so} \quad F(x) > y.$$

We therefore set $x_1 := b - \delta$, and we have shown that there exists x_1 with $a < x_1 < b$ such that $F(x) < y$ in $(x_1,\, b]$.

We can now invoke the Completeness Axiom. We set $c := \sup S$. Since each number $\sup S - 1/n < \sup S$, there exists a sequence $\{s_n\}$ of members of S such that $s_n > \sup S - 1/n$. But then by the Squeeze Theorem,

$$0 < \sup S - s_n < 1/n \quad \text{so} \quad \lim_{n\to\infty} s_n = \sup S = c.$$

We will now use the continuity of $F(x)$ for the second time: since F is continuous on $[a,\, b]$, and since all the s_n are in $S \subseteq [a,\, b]$, we have $\lim_{n\to\infty} F(s_n) = F\left(\lim_{n\to\infty} s_n\right) = F(\sup S) = F(c)$. Also, since $F(s_n) < y$ for each n, we know that $F(c) \leq y$. To check this, we use continuity for the third time, and we suppose not. Then $F(c) > y$. We set $\epsilon := F(c) - y$. Then there exists $\delta > 0$ such that $|x - c| < \delta$ implies that $x \in [a, b]$, *and* that $|F(x) - F(c)| < \epsilon$. By the Archimedean Property, there exists $n \in \mathbb{N}_1$ such that $1/n < \delta$. Then $|s_n - c| < 1/n < \delta$, so

$$F(s_n) - F(c) > -\epsilon = y - F(c), \quad \text{which implies } F(s_n) > y,$$

a contradiction. Thus $F(c) \leq y$.

Next we will show that $F(c) \geq y$. We notice that if $x \in (c, b]$ then $x \notin S$. This is so because $x > c = \sup S$. But then $F(x) \geq y$, since otherwise we would have (i) $x \in [a, b]$ and (ii) $F(x) < y$, which would imply $x \in S$. Now we choose $N \in \mathbb{Z}^+$ such that $1/N < b - c$ (Archimedes!). Then for all $n \in \mathbb{N}$ such that $n \geq N$, $c + 1/n \in [a, b]$ and $F(c + 1/n) \geq y$. For the fourth time we apply continuity and conclude that since $c + 1/n \to c$ and F is continuous on $[a, b]$, we have $F(c + 1/n) \to F(c)$. Our fifth application of continuity uses the argument from our third application, with the inequalities reversed: $y \leq F(c + 1/n) \to F(c)$ implies $y \leq F(c)$.

Hence $F(c) = y$. *This completes the proof of the Intermediate Value Theorem.*

We can now continue our work on $f(x) = e^x$. As we noticed earlier, for each positive y there is a unique integer n such that $f(n) \leq y < f(n+1)$. If $y = f(n)$ we have a solution of our equation. Otherwise, $f(n) < y < f(n+1)$. We can apply the *IVT* (short for Intermediate Value Theorem) since $f(x)$ is continuous, so we find c, $n < c < n + 1$, such that $f(c) = y$. This number c that we obtained using the *IVT* is a solution of the equation $f(x) = y$.

One step remains, and that is to check that the solution is unique. If we had $f(x_1) = y = f(x_2)$ then $x_1 < x_2$ would imply that $y = f(x_1 + (x_2 - x_1)) = f(x_1)f(x_2 - x_1) = yf(x_2 - x_1)$. But since $x_2 > x_1$, $f(x_2 - x_1) > 1$. This gives a contradiction. For the same reasons we cannot have $x_1 > x_2$ either, so $x_1 = x_2$, which is what we mean by "a unique solution."

We have proved that for every $y > 0$ there is a unique real x such that $e^x = y$.

From now on (mostly) we'll write $e^x := f(x)$. Also, for the unique solution x of the equation $f(x) = y$ (> 0) we will write $\log y := x$. Therefore

$$e^{\log y} = y \quad \textit{for all positive real numbers } y. \tag{2.1}$$

We can also write

$$\log e^x = x \quad \textit{for all real numbers } x. \tag{2.2}$$

To see why this is true, we start with a given real x_o and define $y := e^{x_o}$. Then there is a unique real x such that $e^x = y$. Since we already know ("by construction") that $e^{x_o} = y$, we must have $\log(e^{x_o}) = \log y = x = x_o$.

The equations (2.1) and (2.2) express the fact that the functions e^x and $\log y$ are *inverses* of each other: each one undoes the effect of the other. We can expect that properties of each should be "reflected" in properties of the other. It will be hard to stay awake for the development of these basic properties!

Some basic properties of the logarithm function

Two special values of $\log y$ are important to know:

$$\log 1 \;\; \textit{is the solution of the equation} \;\; e^{\log 1} = 1 = f(\log 1). \;\; \textit{Since} \;\; f(0) = 1, \;\; \textit{we know} \;\; \log 1 = 0.$$

We defined $e := f(1)$. Thus $e = e^1$. But we have also defined $\log e$ to be the unique solution x of the equation $e^x = e$. Since $x = 1$ is a solution, we know

$$\log e = 1 \quad \textit{and} \quad \log 1 = 0. \tag{3.1}$$

We know that $e^{u+v} = e^u e^v$. Since we can let $s = e^u$ and $t = e^v$, we have $st = e^{u+v}$, so $\log st = u + v$. But $u = \log s$ and $v = \log t$, so $\log st = \log s + \log t$. This gives us one property:

$$\textit{For all positive numbers } y_1 \textit{ and } y_2, \quad \log y_1 y_2 = \log y_1 + \log y_2. \tag{3.2}$$

Two related formulas:

$$\log(1/y) = -\log y \quad \textit{and} \quad \log(y_1/y_2) = \log y_1 - \log y_2.$$

For $0 = \log 1 = \log y/y = \log y(1/y) = \log y + \log(1/y)$.

Property (3.2) was so easy to verify, proving it felt like just using names! Some other basic properties will require more development:

(3.3) *The function* $\log y$, *defined for all* $y > 0$, *is strictly increasing.*

To prove strict increase, let us consider $0 < y_1 < y_2 < +\infty$. We will begin by showing that for all $r > 0$, $\log r > 0$ if and only if $r > 1$. Then we'll apply this to $r := y_2/y_1$. Now $\log r > 0$ if and only if $e^{\log r} > e^0$, since e^x is strictly increasing. But then we have $\log r > 0$ if and only if $r = e^{\log r} > e^0 = 1$, and this is what we wanted to prove.

To apply the result we just proved, we let $r := y_2/y_1$. Then $r > 0$, so we know $\log y_2 - \log y_1 = \log(y_2/y_1) > 0$ if and only if $y_2/y_1 > 1$. Since $y_1 > 0$ this means exactly that $y_2 > y_1$ if and only if $\log y_2 > \log y_1$.

We will state the next two properties prove them a little later.

(3.4) *The function* $\log y$, *defined for all* $y > 0$, *is continuous and takes on all real values.*

(3.5) *The function* $\log y$ *is differentiable and* $\dfrac{d}{dy}\log y = \dfrac{1}{y}$.

We can use the logarithm to invent a way of raising a positive number b that we can call a "base," to any real power x. In this way we can finally show that the exponent rule $(b^x)^y = b^{xy}$. But you may feel it is all "being done with mirrors." If so, you are right. The only "humanly understandable" ideas about raising to a power are those associated with integer exponents. We can then tack on "roots," $y^{1/n}$, and we have done so. It also makes good sense to talk about $x^{2/3}$; we simply square a cube root. But what does e^π "mean?" The simple idea is to use the exponential function to force the idea of exponent to work, and let it go at that.

Definition of b^x, **when** x **is a real number and** $b > 0$ We know tha since $b > 0$,

$$b = e^{\log b}, \quad \text{so } \textit{by fiat} \text{ we } \textbf{define } b^x := e^{x \log b}. \tag{4.1}$$

An important point about this definition is that since $b^x := e^{x\log b}$, the definition of logarithms yields the formula: $\log b^x = x \log b$.

We now must check that all the properties of exponents that we might expect to be true are true. In addition, we want to know about continuity and about derivative formulas.

The first rule to check is the motivating one: do we have $(b^x)^y = b^{xy}$? This is routine: by definition, $b^{xy} = e^{xy\log b}$. Also by definition, $x \log b = \log b^x$. Thus $xy \log b = y \log b^x$. We now think of b^x as a new "base," and then by definitions $y \log b^x = \log(b^x)^y$. Thus

$$b^{xy} = e^{xy\log b} = e^{y \log b^x} = (b^x)^y. \tag{4.2}$$

Does the rule $b^{x+y} = b^x b^y$ hold true?

$$b^{x+y} = e^{(x+y)\log b} = e^{x\log b + y \log b} = e^{x\log b}e^{y\log b} = b^x b^y,$$

so the rule holds. We had to *recognize* that $e^{x \log b} = b^x$ and $e^{y\log b} = b^y$ by definition.

The remaining questions have to do with continuity and differentiability. We will first answer these questions for $\log y$.

The continuity of $\log y$

(5.1) **Theorem:** *The function* $\log y$, *defined for all* $y > 0$, *is continuous and takes on all real values.*

Proof: The equation $\log e^x = x$ shows that the logarithm function takes on all real values. It remains to *prove* that $\log y$ is continuous. We let $y_o > 0$. We have in mind to look at values y that are close to y_o. Then

$$\log y - \log y_o = \log(y/y_o).$$

We want $\log(y/y_o)$ to be close to zero. That corresponds nicely to having y/y_o be close to 1. That's the *idea.*

Here is another *idea.* Since we have $e^x = y$ if and only if $x = \log y$, to show that $|\log y - \log y_o|$ is small when $|y - y_o|$ is small amounts to showing that $|x - x_o|$ is small when $|e^x - e^{x_o}|$ is small. *The order of the implication is the reverse of the usual implication in continuity.* It is convenient to work with $x_o = 0$, that is, $y_o = 1$.

We know that for $x > 0$, $e^x = f(x) > f_n(x) \geq f_1(x) = 1 + x$. This deserves emphasis:

we have shown that, if $x > 0$ then $e^x > 1 + x$, or, that $e^x - 1 > x$.

Thus $e^x - 1 > x$ when $x > 0$. What about when $x < 0$? Let's work with $-x$, where $x > 0$, to handle negative values of the variable. We have $e^{-x} < 1$, so

$$|e^{-x} - 1| = 1 - e^{-x} = 1 - \frac{1}{e^x} = \frac{e^x - 1}{e^x} > \frac{(x+1) - 1}{e^x} = \frac{x}{e^x} > \frac{x}{e^1} = |-x|/e \text{ if } 0 < x < 1.$$

We can now estimate $|e^x - 1|$ from below if $|x| < 1$:

$$|x| < 1 \Rightarrow |e^x - 1| > |x|/e \tag{5.2}$$

even though this estimate is "too small (smaller than it really needs to be)" when $x > 0$.

Now we can go to work on the "log side." Suppose that $0 < y$ and $0 < y_o$. Then (since differences of x values correspond to ratios of y values) we want $\log(y/y_o)$ to be close to zero when y/y_o is close to one. Thus we put $x := \log(y/y_o)$ and, to use (5.2), we want $|x| = |\log(y/y_o)| < 1$. In other words, $-1 < \log(y/y_o) < 1$, which means $-1 + \log y_o < \log y < \log y_o + 1$. But $-1 = \log(1/e)$ and $1 = \log e$ so we require of y that $\log(y_o/e) < \log y < \log(ey_o)$. This just means (when we apply the strictly increasing function e^x to the inequality we're working on)

$$y_o/e < y < ey_o.$$

If $y_o/e < y < ey_o$ is true then by (5.2) we know that

$$\left|e^{\log y/y_o} - 1\right| > |\log y/y_o|/e.$$

Simplifying gives: If $y_o/e < y < ey_o$ then

$$\frac{|y - y_o|}{y_o} = \left|\frac{y}{y_o} - 1\right| = \left|e^{\log y/y_o} - 1\right| > |\log y/y_o|/e = |\log y - \log y_o|/e, \quad \text{or}$$

$$\textit{if } y_o/e < y < ey_o \textit{ then } |\log y - \log y_o| < e\frac{|y - y_o|}{y_o} = \frac{e}{y_o}|y - y_o|. \tag{5.3}$$

We can now verify, using (5.3) and epsilons and deltas, that $\log y$ is continuous at each $y_o > 0$. The proof of continuity of $\log y$ is complete. Since $\log 1 = 0$, we have in particular that $\log(1 + u) \to 0$ as $u \to 0$.

The differentiability of $\log y$ and the derivative formula $\frac{d}{dy}\log y = 1/y$.

To prove differentiability and obtain the desired formula we need to prove that

$$\textit{for each } y_o > 0, \quad \lim_{k \to 0} \frac{\log(y_o + k) - \log y_o}{k} = \frac{1}{y_o}. \tag{6.0}$$

Let us keep in mind that somehow we have to depend on things we know about e^x. We will start with

$$\frac{\log(y_o+k)-\log y_o}{k}-\frac{1}{y_o}=\frac{\log(y_o+k)/y_o}{k}-\frac{1}{y_o}=\frac{1}{y_o}\left(\frac{\log(1+\frac{k}{y_o})}{\frac{k}{y_o}}-1\right).$$

We will be done with the proof if we can show that

$$(6.1)\qquad \frac{\log(1+\frac{k}{y_o})}{\frac{k}{y_o}}-1\to 0 \ \text{ as } \ k\to 0.$$

Since $k\to 0$ if and only if $\frac{k}{y_o}\to 0$, all we have to do to prove (6.1) is to show that (thinking of $k/y_o =: u$)

$$(6.1a)\qquad \frac{\log(1+u)}{u}-1\to 0 \ \text{ as } \ u\to 0.$$

We have to use e^x somehow. We will set $x:=\log(1+u)$. Then $e^x=1+u$ and so $u=e^x-1$. When we substitute these into $(6.2a)$ we have

$$(6.1b)\qquad \frac{\log(1+u)}{u}-1=\frac{x}{e^x-1}-1=\frac{x}{e^x-1}\left(1-\frac{e^x-1}{x}\right).$$

In the Exponential Saga, as part of the proof that e^x is differentiable and that $(e^x)'=e^x$, we required that $|h|<1$ and proved an inequality,

ES(6.6): $$\left|\tfrac{e^h-1}{h}-1\right|=\left|\tfrac{f(h)-1}{h}-1\right|\le |h|f(|h|)<|h|f(1)=e|h| \ \text{ if } \ |h|<1.$$

We will use this inequality, slightly rearranged, and with h replaced by x:

$$(6.2)\qquad \textit{if } \ 0<|x|<1, \quad \textit{then} \ \left|\frac{e^x-1}{x}-1\right|<e|x|.$$

We will also use the crucial inequality (5.3), with $y=u+1$ and $y_0=1$, which now becomes

$$(6.3)\qquad \textit{if } \ 1/e<1+u<e \ \ \textit{then} \ \ |\log(1+u)-\log 1|=|\log(1+u)|<e\frac{|(1+u)-1|}{1}=e|u|.$$

To be sure that the inequality in (6.3) is true we need to know that $y_o/e<y<ey_o$, namely that $1/e<1+u<e$.
If we require that $|u|<1/16$, then $15/16<1+u<16/15$.
Since $16/15<2<e$, we have $1/e<15/16<1+u<16/15<e$ whenever $|u|<1/16$.
Therefore with $x=\log(1+u)$ we have, by (6.3), that $|x|=|\log(1+u)|<e|u|$. In particular, since $|u|<1/16$, $|x|<e|u|<e/16<1$. We can thus use (6.2), so $\left|\frac{e^x-1}{x}-1\right|<e|x|$. We will use this twice, once as it is, and once noticing that, since $|x|<e/16$, we have $\left|\frac{e^x-1}{x}-1\right|<e^2/16<16\cdot 17e/16\cdot 100=.17e<1/2$ (see Problem 3). But then $-1/2<\frac{e^x-1}{x}-1<1/2$, so that $1/2<\frac{e^x-1}{x}<3/2$. This gives us something useful in $(6.1b)$:

$$\text{with } \ x=\log u, \quad \text{we have } \ \frac{x}{e^x-1}<2, \ \text{ if } \ |u|<1/16.$$

We are now almost done! We can put what we have learned into $(6.1b)$:

$$(6.4)\qquad \left|\frac{\log(1+u)}{u}-1\right|=\frac{x}{e^x-1}\left|1-\frac{e^x-1}{x}\right|<2\left|1-\frac{e^x-1}{x}\right|<2e|x|<2e^2|u|.$$

By the Squeeze Theorem, $(6.1a)$ is now proven to be true. We have now recovered the logarithm function!

Exponentials and Logarithms to other bases

For each positive number b we have defined $b^x := e^{x\log b}$. We need to ask and answer questions about inverse functions and differentiation formulas.

The derivative of b^x. We use the Chain Rule here without proof right now (because we can do these directly, without using the Chain Rule!):

$$\frac{d}{dx}b^x = \frac{d}{dx}e^{x\log b} = e^{x\log b}\frac{d}{dx}x\log b = b^x \log b.$$

Logarithm, base b and its derivative The logarithm base b is a multiple of the natural logarithm:

$$y = b^x \iff y = e^{x\log b} \iff \log y = x\log b \iff x = \frac{\log y}{\log b}.$$

We thus define:

$$\log_b y := \frac{\log y}{\log b}.$$

The derivative of $\log_b y$ is given by

$$\frac{d}{dy}\log_b y = \frac{1}{y\log b}.$$

Problems:

1) Prove that every positive number y lies in a unique interval of the form $[f(n), f(n+1))$, where $n \in \mathbb{Z}$.

2) For $x > 0$, find $\lim_{h\to 0}\frac{x^h - 1}{h}$. Justify your answer!

3) Prove that $e < 16 \cdot 17/100$, an estimate used in the text, and that $.17e < 1/2$.

4) We know $e = e^1 > 1 + 1 = 2$. Prove that $e > 9/4$ and that $e > 64/27$. Generalize.

5) Use (5.3) and epsilons and deltas to prove that $\log y$ is continuous at each $y_o > 0$.

6) Prove that, for all positive x, and all natural numbers n, that $\frac{x^n}{e^x} < C_n/x$, where C_n is a large constant. Find a specific C_n that works. Use this estimate to find $\lim_{x\to\infty} x^n e^{-x}$. Hint: Use the fact that $e^x = f(x) > f_k(x)$ for all $x > 0$ and for all natural numbers k.

7) Prove that $\lim_{y\to+\infty}\frac{\log y}{y} = 0$. Do not use l'Hospital's Rule unless you prove it too! Hint: Put $y = e^x$.

8) Prove that $\lim_{\substack{y\to 0\\ y>0}}\frac{\log y}{y} = -\infty$. Do not use l'Hospital's Rule unless you prove it too! Hint: Put $y = e^x$.

9) Prove the derivative formulas $\frac{d}{dx}b^x = b^x \log b$ and $\frac{d}{dy}\log_b y = \frac{1}{y\log b}$ directly, using only the definition of derivative, and not the Chain Rule. You may, if you wish, make up and prove your own Lemma to cover easy-to-prove cases of the Chain Rule.

10) Which is larger, e^π, or π^e? Justify your answer *mathematically,* not by using a calculator! Hint: Problems (8) and (9) are relevant, along with the First Derivative Test from Calculus.