

These remarks are here because of a discussion after class the Wednesday before Break! We had this definition:

A *Multiresolution Analysis (MRA)* of $L^2(\mathbb{R})$ is a collection $\{V_j\}$ of subspaces of $L^2(\mathbb{R})$ with these properties:

- (i) Each V_j , $-\infty < j < +\infty$, is a closed subspace of $L^2(\mathbb{R})$;
- (ii) For each j , $V_j \subseteq V_{j+1}$, i.e., the spaces V_j are nested;
- (iii) $\bigcap_j V_j = \{0\}$, and $\overline{\bigcup_j V_j} = L^2(\mathbb{R})$;
- (iv) $f(t) \in V_j \iff f(2t) \in V_{j+1}$;
- (v) There is a function $\varphi(t) \in V_0$ such that $\{\varphi(t-n) : n \in \mathbb{Z}\}$ is an o.n. basis for V_0 .

There are three ways to look at this. First, as a *definition*, it's just a set of five conditions about a collection $\{V_j\}$ of subspaces of $L^2(\mathbb{R})$. If all the statements are true, we say that the collection $\{V_j\}$ is an *MRA*. If even one of them is false, then the collection $\{V_j\}$ is *not* an *MRA*. Second, if we are *given* an *MRA* this means that all five statements are true. We have discussed consequences of a collection $\{V_j\}$ of subspaces of $L^2(\mathbb{R})$ *being* an *MRA*. Third, we can start with a collection $\{V_j\}$ of subspaces of $L^2(\mathbb{R})$ and try to prove that it is (or isn't) an *MRA*.

From the second point of view here is something we can deduce from (i) and (v). From (i) we know that V_0 is a closed subspace of $L^2(\mathbb{R})$. From (v) we then conclude that V_0 , being closed, is the closure of its orthonormal basis $\mathcal{B} := \{\varphi(t-n) : n \in \mathbb{Z}\}$. That is, $V_0 = \overline{\text{span}(\mathcal{B})}$.

Now let's look at the third point of view. If we start with a subspace V_0 of $L^2(\mathbb{R})$ that we know to be closed, we can try to find $\varphi(t) \in V_0$ such that $\{\varphi(t-n) : n \in \mathbb{Z}\}$ is an o.n. basis for V_0 . We know that every closed subspace of $L^2(\mathbb{R})$ has an orthonormal basis, but it might not have *any* that consist of integer translates of a single function! This could be a very hard problem. But there is an easier *variation* of the third viewpoint: we can *create* a collection $\{V_j\}$ of closed subspaces of $L^2(\mathbb{R})$. To do this, we start with a function $\varphi(t) \in L^2(\mathbb{R})$ that has the property that $\{\varphi(t-n) : n \in \mathbb{Z}\}$ is an orthonormal set. We saw that the integer translates of the Box function has this property. Then the simple thing to do is to *define* V_0 to be $\overline{\text{span}(\mathcal{B})}$, where $\mathcal{B} := \{\varphi(t-n) : n \in \mathbb{Z}\}$. This makes (v) true. Of course, if $\varphi(t)$ is not the Box function it might be hard to show that $\{\varphi(t-n) : n \in \mathbb{Z}\}$ is an orthonormal set. This is where (II.2) in the *MRA* notes can be useful. Once we have (v) true we can *define* the spaces V_j . There is in fact just one way to do that. There are lots of ways to express how we do it. Here is one way: for every $j \in \mathbb{Z}$ we *define*

$$V_j := \{f(t) \in L^2(\mathbb{R}) : f(t) = g(2^j t) \text{ and } g \in V_0\}. \text{ VERY loosely, this "says" } V_j(t) = V_0(2^j t).$$

Then we have $f(t) \in V_j \iff f(t) = g(2^j t)$ and $g \in V_0$ and we have $f(2t) = g(2^{j+1}t)$, so "by construction" we have $h(t) := f(2t) \in V_{j+1}$. Finally, if $f(2t) \in V_{j+1}$, so that $f(2t) = g(2^{j+1}t)$ and $g \in V_0$, then $f(t) = g(2^j t)$ and $g \in V_0$, so $f(t) \in V_j$. This makes (iv) true. There is a "convenience" built into (iv) that can cause nuisances.

Each of these spaces V_j is a "copy" of V_0 , in an "adjusted" sort of way. The connection between different V_j 's is nicest when it can be done using an "operator," which is the name we give to a function whose domain is a vector space, especially when the operator is linear. Thus let's define an operator $T_j : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ that is linear, that preserves norms, and is invertible. The definition is:

$$\text{for all } g \in L^2(\mathbb{R}) \text{ let } T_j g(t) := 2^{j/2} g(2^j t).$$

In the following Exercises, the third point of view is used.

(R1) **Exercise:** Verify that for all $j \in \mathbb{Z}$, T_j is linear and that $T_j(V_0) = V_j$.

Let us show that T_j preserves norms:

$$\|T_j f\|^2 = \int |T_j f(t)|^2 dt = \int |2^{j/2} f(2^j t)|^2 dt = 2^j \int |f(t)|^2 d(t/2^j) = \int |f(t)|^2 dt = \|f\|^2.$$

The third and fourth equalities came from the literal substitution $t \mapsto t/2^j$.

(R2) **Exercise:** Given j and ℓ in \mathbb{Z} , complete the equation $T_j(V_\ell) = V_?$, verify that it is true, use it to show that each T_j is invertible, and find T_j^{-1} .

(R3) **Exercise:** Verify that if $g_n \in V_0$ and $g_n \rightarrow g$ in L^2 then $g \in V_0$ (already done above!). Then show that every one of the spaces V_j that we defined is closed.

What all this means is that, starting with the variation on (v) we “constructed” subspaces V_j and showed that (iv) and (i) were true. That’s as far as we can go with the variation on the third point of view applied to a single function $\varphi \in L^2(\mathbb{R})$. We have to use features of the function $\varphi(t)$ to show that (ii) is true. In some cases it is easy to show directly that (iii) is true. What is actually true, though it is hard to prove, is that if we know that all *but* (iii) are true, then (iii) is true!

In case φ is the Box function, V_0 is the subspace of L^2 consisting of all functions in $L^2(\mathbb{R})$ that are constant between each consecutive pair of integers. Note that some “almost verywheres” were left out! When we can describe a function in a precise way, we do so, ignoring the differences due to equality a.e. Thus we say: if $f \in V_0$ we know that $f(t) = c_n$ if $n < t < n + 1$. Then we have

$$(R4) \quad \infty > \|f\|^2 = \int |f(t)|^2 dt = \sum_n \int_n^{n+1} |f(t)|^2 dt = \sum_n \int_n^{n+1} |c_n|^2 dt = \sum_n |c_n|^2 = \|\{c_n\}\|^2.$$

Thus the sequence $\{c_n\} \in \ell^2$. Conversely, if we *start* with a sequence $\{c_n\} \in \ell^2$ we can define a function $f(t)$ by setting $f(t) = c_n$ if $n < t < n + 1$, for each $n \in \mathbb{Z}$. Then we can redo (R4) “backwards” to see that $f \in V_0$. This gives us a one-to-one correspondence $T: \ell^2 \rightarrow V_0$ defined by

$$T\{c_n\}(t) := \sum_n c_n B(t - n), \quad \text{and} \quad \|T\{c_n\}\| = \|\{c_n\}\|.$$

(R5) **Exercise:** Verify that $T^{-1}: V_0 \rightarrow \ell^2$ is given by $f(t) \mapsto \left\{ \int_n^{n+1} f(t) dt \right\}_{n=-\infty}^{\infty}$.

We showed that ℓ^2 is a Hilbert space. Therefore V_0 is a closed subspace of $L^2(\mathbb{R})$. Why?

Once we know this, (v) is true for this example. Then both of (i) and (iv) are true as well.

To show that (ii) is true it is enough (why?!) to show that $V_0 \subseteq V_1$. We need two formulas involving the greatest-integer function, $[x]$. This is defined to be the largest integer that is less than or equal to x . This is also known as the “floor” function. Examples: $[2.1] = 2$, $[2] = 2$, $[-2.1] = -3$. The formulas we need: when $k \in \mathbb{Z}$, $[k/2] = n$ if $k = 2n$ and $[k/2] = n$ if $k = 2n + 1$. In the formulas to come we mean by “ k even” that $k = 2n$ and by “ k odd” that $k = 2n + 1$.

We can write $f \in V_0$ as $f(t) = \sum_n c_n B(t - n)$. Since $B(t - n) = B(2t - 2n) + B(2t - 2n - 1)$ a.e., we can write

$$\begin{aligned} f(t) &= \sum_n c_n B(t - n) = \sum_n c_n (B(2t - 2n) + B(2t - 2n - 1)) \\ &= \sum_n c_n B(2t - 2n) + \sum_n c_n B(2t - 2n - 1) \\ &= \sum_{k \text{ even}} c_{[k/2]} B(2t - k) + \sum_{k \text{ odd}} c_{[k/2]} B(2t - k) \\ &= \sum_k c_{[k/2]} B(2t - k) \in V_1. \quad \text{Why is the last equality true?} \end{aligned}$$

It follows that (ii) is true.

Showing that (iii) is true in the Box-function context was done by Assignment 5 #1 and Special Problem 3.