### Definition of a MRA

(II.1) *A Multiresolution Analysis (MRA) of* $L^2(\mathbb{R})$ *is a collection* $\{V_j\}$ *of subspaces of* $L^2(\mathbb{R})$ *with these properties:*

- *(i)* *Each* $V_j,\ -\infty < j < +\infty,$ *is a closed subspace of* $L^2(\mathbb{R})$;
- *(ii)* *For each* $j$, $V_j \subseteq V_{j+1}$, *i.e., the spaces* $V_j$ *are nested;*
- *(iii)* $\bigcap_j V_j = \{0\}$, *and* $\overline{\bigcup_j V_j} = L^2(\mathbb{R})$;
- *(iv)* $f(t) \in V_j \iff f(2t) \in V_{j+1}$;
- *(v)* *There is a function* $\varphi(t) \in V_0$ *such that* $\{\varphi(t-n) : n \in \mathbb{Z}\}$ *is an o.n. basis for* $V_0$.

Our **objective here** is to begin with the *assumption* that we have a MRA, make some convenient assumptions, and derive some equations that we can use later, "working backwards," to construct a scaling function and a MRA. We will be looking for scaling functions with *compact support,* namely functions $\varphi(t)$ that satisfy (v) above (scaling functions) and that are zero outside some bounded closed interval (the *support* of a function is the smallest *closed* set that contains all the points where the function is not zero). We have one example already: the one with $\varphi$ being the box function and $V_0$ being the space of functions in $L^2(\mathbb{R})$ that are constant on the intervals between consecutive integers.

We will distinguish between two kinds of facts or assumptions: Type I – conditions (i)-(v) in a MRA, and Type II – all others made in connection with a MRA. Some Type II assumptions will follow easily from our search for compactly supported scaling functions.

### How the numbers $h(n)$ are found

We know that (iv) and (v) tell us that the functions $2^{j/2}\varphi(2^j t - k)$, where $k \in \mathbb{Z}$, form an orthonormal (o.n.) basis for $V_j$. By these and by (ii) we thus know that $\varphi \in V_1$, and so there exist constants $h(n)$ so that

(Scaling)
$$\varphi(t) = \sum_n h(n)\sqrt{2}\,\varphi(2t-n), \quad \text{and} \quad \sum_n |h(n)|^2 = 1.$$

These two statements are **Type I deductions.** The first is about a series that makes sense as a series in $L^2$, not necessarily as a series of functions!! **Main Objective: we want equations about these numbers** $h(n)$.

### Four equations about the $h(n)$

We thus have one equation involving the $h(n)$:

($h1$)
$$\sum_n |h(n)|^2 = 1.$$

**Example (box function):** $\varphi(t) = B(t) = B(2t) + B(2t-1) = \frac{1}{\sqrt{2}}\sqrt{2}\varphi(2t) + \frac{1}{\sqrt{2}}\sqrt{2}\varphi(2t-1)$, so we have $h(0) = \frac{1}{\sqrt{2}} = h(1)$ and all other $h(n) = 0$.

We need to use the **Fourier transform on** $L^2$, and these two facts about it:

($L^2$: translation)
$$\widehat{\left(f(t-h)\right)}(\xi) = e^{-i\xi h}\hat{f}(\xi)$$

and

($L^2$: dilation)
$$\widehat{\left(f(\lambda t)\right)}(\xi) = \frac{1}{\lambda}\hat{f}\left(\frac{\xi}{\lambda}\right).$$

These agree with the corresponding facts about $L^1$ Fourier transforms. However, the Fourier transform of an $L^2$ function is not actually gotten from an integral, but by $L^2$-limits of Fourier transforms of $L^1$ functions. More information about this is in the note "Existence of $L^2$ Fourier Transform" linked to the Math 5467 Web page. In case $f$ belongs to both spaces, $f \in L^2 \cap L^1$, their Fourier transforms agree in both senses.

We apply these facts to the equation (Scaling), and obtain

(Fourier Scaling) $$\widehat{\varphi}(\xi) = \sum_n \frac{h(n)}{\sqrt{2}}\, e^{-in\xi/2}\widehat{\varphi}\left(\frac{\xi}{2}\right) =: m_o(\xi/2)\widehat{\varphi}(\xi/2).$$

This equation *defines* the function $m_o(\xi)$ whose formula is (another $L^2$ series, this one in $L^2(\mathbb{T})$)

(Low-pass Filter) $$m_o(\xi) := \sum_n \frac{h(n)}{\sqrt{2}}\, e^{-in\xi}.$$

We will now make our first **Type II Assumption:** $\sum_n |h(n)| < \infty$.

This assumption will be true if only finitely many of the $h(n)$ are non-zero. It follows from this assumption that $m_o(\xi)$ is a continuous function of $\xi$. We'll now make another assumption that makes the equation (Fourier Scaling) into an equation between continuous functions.

Our second **Type II Assumption:** $\varphi \in L^1$. That is, we suppose $\varphi$ is an integrable function.

We know that the Fourier transform of an integrable function is continuous. Thus all the members in (Fourier Scaling) are continuous. We want to set $\xi = 0$ and then cancel $\widehat{\varphi}(0)$, so we make

Our third **Type II Assumption:** $\widehat{\varphi}(0) \neq 0$. That is, we suppose $\int \varphi(t)\, dt \neq 0$, and then we find, from (Fourier Scaling), that $m_o(0) = 1$.

Now we have equation about the $h(n)$:

($h2$) $$1 = m_o(0) = \sum_n \frac{h(n)}{\sqrt{2}} \quad \text{or,} \quad \sum_n h(n) = \sqrt{2}\,.$$

**Example: the Box function** All of the Type II Assumptions hold. A direct computation gives

$$\widehat{\varphi}(\xi) = \widehat{B}(\xi) = e^{-i\xi/2}\frac{\sin \xi/2}{\xi/2} \quad \text{and} \quad m_o(\xi) = \frac{1+e^{-i\xi}}{2} = e^{-i\xi/2}\cos(\xi/2).$$

The next important equation is

$$|m_o(\xi)|^2 + |m_o(\xi+\pi)|^2 = 1 \quad \text{a.e.}$$

Under all our Type II Assumptions so far, this becomes

$$|m_o(\xi)|^2 + |m_o(\xi+\pi)|^2 = 1 \quad \text{for all} \quad \xi.$$

This equation will be derived later. Assuming that it is true, we have, on setting $\xi = 0$ and using $(h1)$, $m_o(\pi) = 0$. When we convert this into the formula for $m_o(\pi)$ and use $e^{\pm i\pi} = -1$, we have

($h3$) $$0 = m_o(\pi) = \sum_n \frac{h(n)}{\sqrt{2}}(-1)^n \quad \text{or,} \quad \sum_n (-1)^n h(n) = 0.$$

**The derivation:** One of our Type I Assumptions about the integer translates of the scaling function $\varphi$ is that they form an orthonormal set. Therefore we have **the orthogonality equation**

$$\delta_{mn} = \langle \varphi(t-m), \varphi(t-n)\rangle = \int \varphi(t-m)\overline{\varphi(t-n)}\, dt = \int \varphi(t-(m-n))\overline{\varphi(t)}\, dt = \delta_{m-n,0},$$

which is convenient: we only have to compare a translate of $\varphi$ to $\varphi$. We will next use

**the Plancherel formula for the Fourier transform on $L^2(\mathbb{R})$:**

$$\langle \hat{f}, \hat{g}\rangle = 2\pi\langle f, g\rangle,$$

applied to the orthogonality equation above:

$$\delta_{k0} = \int \varphi(t-k)\overline{\varphi(t)}\,dt = \frac{1}{2\pi}\int e^{-ik\xi}\widehat{\varphi}(\xi)\overline{\widehat{\varphi}(\xi)}\,dt = \frac{1}{2\pi}\int e^{-ik\xi}|\widehat{\varphi}(\xi)|^2\,dt.$$

Now comes a **periodization argument:**

$$\begin{aligned}
\delta_{k0} &= \frac{1}{2\pi}\int e^{-ik\xi}|\widehat{\varphi}(\xi)|^2\,dt \\
&= \frac{1}{2\pi}\sum_n \int_{2\pi n}^{2\pi(n+1)} e^{-ik\xi}|\widehat{\varphi}(\xi)|^2\,dt \\
&= \frac{1}{2\pi}\sum_n \int_0^{2\pi} e^{-ik(\xi+2\pi n)}|\widehat{\varphi}(\xi+2\pi n)|^2\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\xi}\sum_n |\widehat{\varphi}(\xi+2\pi n)|^2\,dt,
\end{aligned}$$

where we have interchanged sum and integral (which is allowed, by Fubini's Theorem) and used: $e^{2\pi m} = 1$ for all integers $m$. The very last integral is the formula for the Fourier coefficient $G_k$ of the $2\pi$-periodic and integrable (over any interval of length $2\pi$) function

$$G(\xi) := \sum_n |\widehat{\varphi}(\xi+2\pi n)|^2.$$

Since all the Fourier coefficients are zero except the one for $k=0$, and $G_0 = 1$, we have discovered that $G(\xi) = 1$ a.e.

If we now replace $\xi$ in the last equation by $2\xi$, we have (we will exploit the fact that $m_o$ is a $2\pi$-periodic function)

$$\begin{aligned}
1 &= \sum_n |\widehat{\varphi}(2\xi+2\pi n)|^2 \\
&= \sum_n |m_o(\xi+\pi n)|^2|\widehat{\varphi}(\xi+\pi n)|^2 \\
&= \sum_{n \text{ even}} |m_o(\xi+\pi n)|^2|\widehat{\varphi}(\xi+\pi n)|^2 + \sum_{n \text{ odd}} |m_o(\xi+\pi n)|^2|\widehat{\varphi}(\xi+\pi n)|^2 \\
&= \sum_{n \text{ even}} |m_o(\xi)|^2|\widehat{\varphi}(\xi+\pi n)|^2 + \sum_{n \text{ odd}} |m_o(\xi+\pi)|^2|\widehat{\varphi}(\xi+\pi n)|^2 \\
&= \sum_k |m_o(\xi)|^2|\widehat{\varphi}(\xi+\pi(2k))|^2 + \sum_k |m_o(\xi+\pi)|^2|\widehat{\varphi}(\xi+\pi(2k+1))|^2 \\
&= \sum_k |m_o(\xi)|^2|\widehat{\varphi}(\xi+2\pi k)|^2 + \sum_k |m_o(\xi+\pi)|^2|\widehat{\varphi}(\xi+\pi+2\pi)|^2 \\
&= |m_o(\xi)|^2| + |m_o(\xi+\pi)|^2 \quad \text{a.e.}
\end{aligned}$$

There is a Theorem about the "Fourier transform side" of orthonormal integer translates that we essentially proved here, just before the last discussion about $m_o$:

(II.2) **Theorem (orthonormal integer translates):** *The sequence $g(t-n)$ of all integer translates of a function $g \in L^2(\mathbb{R})$ is an orthonormal set if and only if*

$$G(\xi) := \sum_n |\widehat{g}(\xi+2\pi n)|^2 = 1 \quad \text{a.e.}$$

**Conversion of "the orthogonality equation" into an equation about the** $h(n)$

We need to convert the equation $\delta_{k,0} = \langle \varphi(t-k), \varphi(t) \rangle$ into one about the numbers $h(n)$. Here is the calculation, in which we will use the notation $\varphi_{1,\ell}(t) := \sqrt{2}\,\varphi(2t-\ell)$:

$$\begin{aligned}\delta_{k0} &= \int \varphi(t-k)\overline{\varphi(t)}\,dt = \int \sum_n h(n)\varphi_{1,n}(t-k)\overline{\sum_m h(m)\varphi_{1,m}(t)}\,dt \\ &= \sum_n h(n) \sum_m \overline{h(m)} \int \varphi_{1,n}(t-k)\overline{\varphi_{1,m}(t)}\,dt,\end{aligned}$$

and we have

$$\int \varphi_{1,n}(t-k)\overline{\varphi_{1,m}(t)}\,dt = \int \sqrt{2}\,\varphi(2(t-k)-n)\overline{\sqrt{2}\,\varphi(2t-m)}\,dt = 2\int \varphi(2(t-k)-n)\overline{\varphi(2t-m)}\,dt.$$

But the last integral is

$$2\int \varphi(2t-(2k+n))\overline{\varphi(2t-m)}\,dt = \delta_{2k+n,m}$$

so that, going back, we have

$$\begin{aligned}\delta_{k0} &= \sum_n h(n) \sum_m \overline{h(m)} \int \varphi_{1,n}(t-k)\overline{\varphi_{1,m}(t)}\,dt \\ &= \sum_n h(n) \sum_m \overline{h(m)}\delta_{2k+n,m} \\ &= \sum_n h(n)\overline{h(2k+n)}.\end{aligned}$$

After a minor change we have our next equation about the $h(n)$.

$(h4)$
$$\sum_n h(n)\overline{h(n+2k)} = \delta_{k0} = \begin{cases} 1, & \text{if } k=0; \\ 0, & \text{if } k \neq 0. \end{cases}$$

Actually, $(h1)$ is included, as the case $k=0$, in $(h4)$. Note that the translate of the $h(n)$ that appears in $(h4)$ is *even*.

(II.3) **Summary so far:**

$(h1)$
$$\sum_n |h(n)|^2 = 1.$$

$(h2)$
$$1 = m_o(0) = \sum_n \frac{h(n)}{\sqrt{2}} \quad \text{or,} \quad \sum_n h(n) = \sqrt{2}.$$

$(h3)$
$$0 = m_o(\pi) = \sum_n \frac{h(n)}{\sqrt{2}}(-1)^n \quad \text{or,} \quad \sum_n (-1)^n h(n) = 0.$$

$(h4)$
$$\sum_n h(n)\overline{h(n+2k)} = \delta_{k0} = \begin{cases} 1, & \text{if } k=0; \\ 0, & \text{if } k \neq 0. \end{cases}$$

**The Cascade Formula**

We can iterate in the (Fourier Scaling) equation. If we do so $N$ times we get

(Iterated Fourier Scaling)
$$\widehat{\varphi}(\xi) = \left(\prod_{k=1}^{N-1} m_o(\xi/2^k)\right)\widehat{\varphi}(\xi/2^N).$$

Under all our assumptions of Type II, as $N \to \infty$ the factor $\widehat{\varphi}(\xi/2^N)$ converges to a non-zero limit. Therefore the products converge as well, to an infinite product, and we have *The Cascade Formula:*

(Cascade Formula)
$$\widehat{\varphi}(\xi) = \widehat{\varphi}(0)\prod_{k=1}^{\infty} m_o(\xi/2^k).$$

There is a useful Theorem in our case: $|\widehat{\varphi}(0)| = 1$! By choosing an appropriate constant, we can (and do!) assume that $\widehat{\varphi}(0) = 1$.

First we will **review.**

We began with a MRA and introduced the *scaling equation,*

(Scaling)
$$\varphi(t) = \sum_n h(n)\sqrt{2}\,\varphi(2t-n), \quad \text{and} \quad \sum_n |h(n)|^2 = 1.$$

and its Fourier transform version

(Fourier Scaling)
$$\widehat{\varphi}(\xi) = \sum_n \frac{h(n)}{\sqrt{2}}\, e^{-in\xi/2}\widehat{\varphi}\left(\frac{\xi}{2}\right).$$

We used (Fourier Scaling) to define the *low-pass filter,* a function $m_o(\xi)$ whose formula is (another $L^2$ series, this one in $L^2(\mathbb{T})$):

(Low-pass Filter)
$$m_o(\xi) := \sum_n \frac{h(n)}{\sqrt{2}}\, e^{-in\xi}.$$

We then used the orthonormality of $\{\varphi(t-n) : n \in \mathbb{Z}\}$, the integer translates of the *scaling function* $\varphi$, to show that the Fourier transform of the scaling function satisfies

$$\sum_n |\widehat{\varphi}(\xi + 2\pi n)|^2 = 1 \quad \text{a.e.,}$$

And then we showed that the low-pass filter $m_o$ satisfies

$$|m_o(\xi)|^2| + |m_o(\xi+\pi)|^2 = 1 \quad \text{a.e.}$$

We used some special assumptions we called Type II assumptions to show that $m_o$ is continuous, so it is actually true (for us) that

$$|m_o(\xi)|^2| + |m_o(\xi+\pi)|^2 \equiv 1.$$

Two of our Type II assumptions allowed us to show that $m_o(0) = 1$, and we went on to show that the $h(n)$ satisfy these conditions:

($h1$)
$$\sum_n |h(n)|^2 = 1.$$

($h2$)
$$1 = m_o(0) = \sum_n \frac{h(n)}{\sqrt{2}} \quad \text{or,} \quad \sum_n h(n) = \sqrt{2}\,.$$

(h3)
$$0 = m_o(\pi) = \sum_n \frac{h(n)}{\sqrt{2}}(-1)^n \quad \text{or,} \quad \sum_n (-1)^n h(n) = 0.$$

(h4)
$$\sum_n h(n)\overline{h(n-2k)} = \delta_{k0} = \begin{cases} 1, & \text{if } k = 0; \\ 0, & \text{if } k \neq 0. \end{cases}$$

Using all this we found the *The Cascade Formula:*

(Cascade Formula)
$$\widehat{\varphi}(\xi) = \widehat{\varphi}(0) \prod_{k=1}^{\infty} m_o(\xi/2^k).$$

Here are our three Type II assumptions:

**Type II Assumption 1:** $\sum_n |h(n)| < \infty$.

**Type II Assumption 2:** $\varphi \in L^1$. That is, we suppose $\varphi$ is an integrable function.

**Type II Assumption 3:** $\widehat{\varphi}(0) \neq 0$. That is, we suppose $\int \varphi(t)\,dt \neq 0$.

**end of review**

**Type II Assumption 4:** $\varphi$ *has compact support.*

That is, there is a closed and bounded set, outside of which $\varphi(t) \equiv 0$. If we don't need to know exactly which set is the *support* of $\varphi$, we can just say there is a positive number $M$ such that $\varphi(t) \equiv 0$ if $|t| > M$.

We will use the new Type II assumption to show that only finitely many of the $h(n)$ can be non-zero. Let us take another look at the Scaling Equation. All we have used so far is that, since $V_0 \subseteq V_1$, and since $\varphi \in V_0$, and $\|\varphi\| = 1$, we know that

$$\varphi(t) = \sum_n h(n)\sqrt{2}\,\varphi(2t-n), \quad \text{and that} \quad \sum_n |h(n)|^2 = 1.$$

However, we know that this representation, as a series in $L^2$, is unique, so by the Projection Theorem we actually know that

(II.4)
$$\varphi = P_{V_1}\varphi = \sum_n \left\langle \varphi,\, \sqrt{2}\,\varphi(2s-n)\right\rangle_{ds} \sqrt{2}\,\varphi(2t-n),$$

and so

$$h(n) = \left\langle \varphi,\, \sqrt{2}\,\varphi(2s-n)\right\rangle_{ds} = \int \varphi(s)\overline{\sqrt{2}\,\varphi(2s-n)}\,ds.$$

Now, to show that all but finitely many of the $h(n)$ are zero, we look at the integral defining $h(n)$. In order that $h(n) \neq 0$ it *has* to be true that the integrand is not identically zero. Of course, that by itself does not guarantee that $h(n) \neq 0$, it's just a necessary condition. We will be "crude" about the support and just say that $\varphi(t) = 0$ if $|t| > M$, for some positive $M$. But then to have the integrand be non-zero, it has to be true that, for some $s$, $|s| \leq M$ and that (at the same time) $|2s - n| \leq M$. This means that, simultaneously,

$$-M \leq s \leq M \quad \text{and} \quad \frac{n-M}{2} \leq s \leq \frac{n-M}{2}\,.$$

Now if $n \geq 3M$, we'd have $s \leq M = \frac{3M-M}{2} \leq \frac{n-M}{2} \leq s$, which would give an integrand non-zero at at most one point and hence a zero $h(n)$. Thus $n < 3M$ has to be true if $h(n) \neq 0$. This argument can be "multiplied by $-1$." to show finally that $h(n) = 0$ unless $|n| < 3M$.

Some things follow from this! Type II Assumption 1, that $\sum_n |h(n)| < \infty$, is automatically true if Type II Assumption 4 is true. We can also show that Type II Assumption 2, that $\varphi \in L^1$, is automatically true if Type II Assumption 4 is true, as follows.

With $\varphi \in L^2$ and zero if $|t| > M$, the Schwarz inequality gives

$$\int |\varphi(t)|\,dt = \int [|t| \leq M]|\varphi(t)|\,dt \leq \left( \int [|t| \leq M]^2 \right)^{1/2} \left( \int |\varphi(t)|^2 \right)^{1/2} = (2M)^{1/2}\,\|\varphi\|_2 < \infty.$$

We can put this into the form of a Theorem.

(II.5) **Theorem:** *If $\varphi$ is in $L^2$ and has compact support, and if the conditions* (i) − (v) *of a MRA are true for $\varphi$ then $\varphi \in L^1$, (that is $\varphi$ is integrable). If, in addition, $\int \varphi(t)\,dt \neq 0$, then the low-pass filter, $m_o$, is a trigonometric polynomial, i.e. it has only finitely many non-zero coefficients $h(n)$, so that $\sum_n |h(n)| < \infty$.*