

**Definition of an MRA**

(II.1) A *Multiresolution Analysis (MRA)* of  $L^2(\mathbb{R})$  is a collection  $\{V_j\}$  of subspaces of  $L^2(\mathbb{R})$  with these properties:

- (i) Each  $V_j$ ,  $-\infty < j < +\infty$ , is a closed subspace of  $L^2(\mathbb{R})$ ;
- (ii) For each  $j$ ,  $V_j \subseteq V_{j+1}$ , i.e., the spaces  $V_j$  are nested;
- (iii)  $\bigcap_j V_j = \{0\}$ , and  $\overline{\bigcup_j V_j} = L^2(\mathbb{R})$ ;
- (iv)  $f(t) \in V_j \iff f(2t) \in V_{j+1}$ ;
- (v) There is a function  $\varphi(t) \in V_0$  such that  $\{\varphi(t-n) : n \in \mathbb{Z}\}$  is an o.n. basis for  $V_0$ .

Our **objective here** is to begin with the *assumption* that we have an MRA, make some convenient assumptions, and derive some equations (Low-pass Filter Equations) that we can use later, “working backwards,” to *construct* a scaling function and an MRA. We will be looking for scaling functions with *compact support*, namely functions  $\varphi(t)$  that satisfy (v) above (scaling functions) and that are zero outside some bounded closed interval (the *support* of a function is the smallest *closed* set that contains all the points where the function is not zero). We have one example already: when  $\varphi$  is the box function and  $V_0$  is the space of functions in  $L^2(\mathbb{R})$  that are constant on the intervals between consecutive integers.

We will distinguish between two kinds of facts or assumptions: Type I – conditions (i)-(v) in an MRA, and Type II – all others made in connection with an MRA. Some Type II assumptions will arise from our search for compactly supported scaling functions.

**How certain important numbers  $h(n)$  are found**

We know that (iv) and (v) tell us that the functions  $2^{j/2}\varphi(2^j t - k)$ , where  $k \in \mathbb{Z}$ , form an orthonormal (o.n.) basis for  $V_j$ . By these and by (ii) we know that  $\varphi \in V_1$ , so there exist constants  $h(n)$  so that

(Scaling) 
$$\varphi(t) = \sum_n h(n)\sqrt{2} \varphi(2t - n), \text{ and } \sum_n |h(n)|^2 = 1.$$

These two statements are **Type I deductions**. The first is about a series that makes sense as a series in  $L^2$ , not necessarily as a series of functions!! **A Main Objective: we want equations about these numbers  $h(n)$** . The equations will be called **Low-pass Filter Equations**

**Four equations about the  $h(n)$**

So far we have one equation involving the  $h(n)$ :

(h1) 
$$\sum_n |h(n)|^2 = 1.$$

**Example (box function):**  $\varphi(t) = B(t) = B(2t) + B(2t - 1) = \frac{1}{\sqrt{2}}\sqrt{2}\varphi(2t) + \frac{1}{\sqrt{2}}\sqrt{2}\varphi(2t - 1)$ , so we have  $h(0) = \frac{1}{\sqrt{2}} = h(1)$  and all other  $h(n) = 0$ .

We need to use the **Fourier transform on  $L^2$** , and these two facts about it:

( $L^2$ : translation) 
$$(f(t-h))^\wedge(\xi) = e^{-i\xi h} \hat{f}(\xi)$$

and

( $L^2$ : dilation) 
$$(f(\lambda t))^\wedge(\xi) = \frac{1}{\lambda} \hat{f}\left(\frac{\xi}{\lambda}\right).$$

These agree with the corresponding facts about  $L^1$  Fourier transforms. However, the Fourier transform of an  $L^2$  function is not actually gotten from an integral, but by  $L^2$ -limits of Fourier transforms of  $L^1$  functions. More

information about this is in the note “Existence of  $L^2$  Fourier Transform” linked to the Math 5467 Web page. In case  $f$  belongs to both spaces,  $f \in L^2 \cap L^1$ , their Fourier transforms agree in both senses.

We apply these facts to the equation (Scaling), and obtain

$$(Fourier\ Scaling) \quad \widehat{\varphi}(\xi) = \sum_n \frac{h(n)}{\sqrt{2}} e^{-in\xi/2} \widehat{\varphi}\left(\frac{\xi}{2}\right) =: m_o(\xi/2)\widehat{\varphi}(\xi/2).$$

This equation defines the function  $m_o(\xi)$  whose formula is (another  $L^2$  series, this one in  $L^2(\mathbb{T})$ )

$$(Low-pass\ Filter) \quad m_o(\xi) \sim \sum_n \frac{h(n)}{\sqrt{2}} e^{-in\xi}.$$

We will now make our first **Type II Assumption**:  $\sum_n |h(n)| < \infty$ . This implies  $m_o(\xi) = \sum_n \frac{h(n)}{\sqrt{2}} e^{-in\xi}$ .

This assumption will be true if only finitely many of the  $h(n)$  are non-zero. It follows from this assumption that  $m_o(\xi)$  is a continuous function of  $\xi$ . We’ll now make another assumption that makes the equation (Fourier Scaling) into an equation between continuous functions.

Our second **Type II Assumption**:  $\varphi \in L^1$ . That is, we suppose  $\varphi$  is an integrable function.

We know that the Fourier transform of an integrable function is continuous. Thus all the members in (Fourier Scaling) are continuous. We want to set  $\xi = 0$  and then cancel  $\widehat{\varphi}(0)$ , so we make

Our third **Type II Assumption**:  $\widehat{\varphi}(0) \neq 0$ . That is, we suppose  $\int \varphi(t) dt \neq 0$ , and then we find, from (Fourier Scaling), that  $m_o(0) = 1$ .

Now we have another equation about the  $h(n)$ :

$$(h2) \quad 1 = m_o(0) = \sum_n \frac{h(n)}{\sqrt{2}} \quad \text{or,} \quad \sum_n h(n) = \sqrt{2}.$$

**Example: the Box function** All of the Type II Assumptions hold. A direct computation gives

$$\widehat{\varphi}(\xi) = \widehat{B}(\xi) = e^{-i\xi/2} \frac{\sin \xi/2}{\xi/2} \quad \text{and} \quad m_o(\xi) = \frac{1+e^{-i\xi}}{2} = e^{-i\xi/2} \cos(\xi/2).$$

The next important equation is

$$|m_o(\xi)|^2 + |m_o(\xi + \pi)|^2 = 1 \quad \text{a.e.}$$

Under all our Type II Assumptions so far, this becomes

$$|m_o(\xi)|^2 + |m_o(\xi + \pi)|^2 = 1 \quad \text{for all } \xi.$$

This equation will be derived soon. Assuming that it is true, we have, on setting  $\xi = 0$  and using (h2),  $m_o(\pi) = 0$ . When we convert this into the series formula for  $m_o(\pi)$  and use  $e^{\pm i\pi} = -1$ , we have

$$(h3) \quad 0 = m_o(\pi) = \sum_n \frac{h(n)}{\sqrt{2}} (-1)^n \quad \text{or,} \quad \sum_n (-1)^n h(n) = 0.$$

**The derivation**: One of our Type I Assumptions about the integer translates of the scaling function  $\varphi$  is that they form an orthonormal set. Therefore we have **the orthogonality equation**

$$\delta_{mn} = \langle \varphi(t - m), \varphi(t - n) \rangle = \int \varphi(t - m) \overline{\varphi(t - n)} dt = \int \varphi(t - (m - n)) \overline{\varphi(t)} dt = \delta_{m-n,0},$$

which is convenient: we only have to compare a translate of  $\varphi$  to  $\varphi$ . We will next use

**the Plancherel formula for the Fourier transform on  $L^2(\mathbb{R})$**  ((12), (15) and (15’) in *Fourier Facts:  $L^2$* ):

$$\langle \widehat{f}, \widehat{g} \rangle = 2\pi \langle f, g \rangle,$$

applied to the orthogonality equation above:

$$\delta_{k0} = \int \varphi(t-k)\overline{\varphi(t)} dt = \frac{1}{2\pi} \int e^{-ik\xi} \widehat{\varphi}(\xi) \overline{\widehat{\varphi}(\xi)} d\xi = \frac{1}{2\pi} \int e^{-ik\xi} |\widehat{\varphi}(\xi)|^2 d\xi.$$

Now comes a **periodization argument**:

$$\begin{aligned} \delta_{k0} &= \frac{1}{2\pi} \int e^{-ik\xi} |\widehat{\varphi}(\xi)|^2 d\xi \\ &= \frac{1}{2\pi} \sum_n \int_{2\pi n}^{2\pi(n+1)} e^{-ik\xi} |\widehat{\varphi}(\xi)|^2 d\xi \\ &= \frac{1}{2\pi} \sum_n \int_0^{2\pi} e^{-ik(\xi+2\pi n)} |\widehat{\varphi}(\xi+2\pi n)|^2 d\xi \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\xi} \sum_n |\widehat{\varphi}(\xi+2\pi n)|^2 d\xi, \end{aligned}$$

where we have interchanged sum and integral (which is allowed, by Fubini's Theorem) and used:  $e^{2\pi im} = 1$  for all integers  $m$ . The very last integral is the formula for the Fourier coefficient  $G_k$  of the  $2\pi$ -periodic and integrable (over any interval of length  $2\pi$ ) function

$$G(\xi) := \sum_n |\widehat{\varphi}(\xi+2\pi n)|^2.$$

Since all the Fourier coefficients,  $G_k$ , of  $G(\xi)$  are zero except the one for  $k=0$ , and  $G_0=1$ , we have discovered that  $G(\xi) = 1$  a.e.

If we now replace  $\xi$  in the last equation by  $2\xi$ , we have (we will exploit the fact that  $m_o$  is a  $2\pi$ -periodic function)

$$\begin{aligned} 1 &= \sum_n |\widehat{\varphi}(2\xi+2\pi n)|^2 \\ &= \sum_n |m_o(\xi+\pi n)|^2 |\widehat{\varphi}(\xi+\pi n)|^2 \\ &= \sum_{n \text{ even}} |m_o(\xi+\pi n)|^2 |\widehat{\varphi}(\xi+\pi n)|^2 + \sum_{n \text{ odd}} |m_o(\xi+\pi n)|^2 |\widehat{\varphi}(\xi+\pi n)|^2 \\ &= \sum_{n \text{ even}} |m_o(\xi)|^2 |\widehat{\varphi}(\xi+\pi n)|^2 + \sum_{n \text{ odd}} |m_o(\xi+\pi)|^2 |\widehat{\varphi}(\xi+\pi n)|^2 \\ &= \sum_k |m_o(\xi)|^2 |\widehat{\varphi}(\xi+\pi(2k))|^2 + \sum_k |m_o(\xi+\pi)|^2 |\widehat{\varphi}(\xi+\pi(2k+1))|^2 \\ &= \sum_k |m_o(\xi)|^2 |\widehat{\varphi}(\xi+2\pi k)|^2 + \sum_k |m_o(\xi+\pi)|^2 |\widehat{\varphi}(\xi+\pi+2\pi k)|^2 \\ &= |m_o(\xi)|^2 + |m_o(\xi+\pi)|^2 \text{ a.e.} \end{aligned}$$

There is a Theorem about the "Fourier transform side" of orthonormal integer translates that we essentially proved here, just before the last discussion about  $m_o$ :

**(II.2) Theorem (orthonormal integer translates):** *The sequence  $g(t-n)$  of all integer translates of a function  $g \in L^2(\mathbb{R})$  is an orthonormal set if and only if*

$$G(\xi) := \sum_n |\widehat{g}(\xi+2\pi n)|^2 = 1 \text{ a.e.}$$

**Conversion of “the orthogonality equation” into an equation about the  $h(n)$**

We need to convert the equation  $\delta_{k,0} = \langle \varphi(t-k), \varphi(t) \rangle$  into one about the numbers  $h(n)$ . Here is the calculation, in which we will use the notation  $\varphi_{1,\ell}(t) := \sqrt{2} \varphi(2t-\ell)$ :

$$\begin{aligned}
 \delta_{k0} &= \int \varphi(t-k) \overline{\varphi(t)} dt = \int \sum_n h(n) \varphi_{1,n}(t-k) \overline{\sum_m h(m) \varphi_{1,m}(t)} dt \\
 (*) \quad &= \sum_n h(n) \sum_m \overline{h(m)} \int \varphi_{1,n}(t-k) \overline{\varphi_{1,m}(t)} dt,
 \end{aligned}$$

and we have

$$\int \varphi_{1,n}(t-k) \overline{\varphi_{1,m}(t)} dt = \int \sqrt{2} \varphi(2(t-k)-n) \overline{\sqrt{2} \varphi(2t-m)} dt = 2 \int \varphi(2(t-k)-n) \overline{\varphi(2t-m)} dt.$$

But this equation tells us that

$$\int \varphi_{1,n}(t-k) \overline{\varphi_{1,m}(t)} dt = \int \sqrt{2} \varphi(2t-(2k+n)) \overline{\sqrt{2} \varphi(2t-m)} dt = \delta_{2k+n,m}.$$

Hence, going back to (\*), we have

$$\begin{aligned}
 \delta_{k0} &= \sum_n h(n) \sum_m \overline{h(m)} \int \varphi_{1,n}(t-k) \overline{\varphi_{1,m}(t)} dt \\
 &= \sum_n h(n) \sum_m \overline{h(m)} \delta_{2k+n,m} \\
 &= \sum_n h(n) \overline{h(2k+n)}.
 \end{aligned}$$

After a minor change we have our next equation about the  $h(n)$ .

$$(h4) \quad \sum_n h(n) \overline{h(n+2k)} = \delta_{k0} = \begin{cases} 1, & \text{if } k = 0; \\ 0, & \text{if } k \neq 0. \end{cases}$$

Actually, (h1) is included, as the case  $k = 0$ , in (h4). Note that the index of translation of the  $h(n)$  that appears in (h4) is even.

**(II.3) Summary so far: the first four Low-pass Filter Equations**

$$(h1) \quad \sum_n |h(n)|^2 = 1.$$

$$(h2) \quad 1 = m_o(0) = \sum_n \frac{h(n)}{\sqrt{2}} \quad \text{or,} \quad \sum_n h(n) = \sqrt{2}.$$

$$(h3) \quad 0 = m_o(\pi) = \sum_n \frac{h(n)}{\sqrt{2}} (-1)^n \quad \text{or,} \quad \sum_n (-1)^n h(n) = 0.$$

$$(h4) \quad \sum_n h(n) \overline{h(n+2k)} = \delta_{k0} = \begin{cases} 1, & \text{if } k = 0; \\ 0, & \text{if } k \neq 0. \end{cases}$$

**The Cascade Formula**

We can iterate in the (Fourier Scaling) equation. If we do so  $N$  times we get

(Iterated Fourier Scaling) 
$$\widehat{\varphi}(\xi) = \left( \prod_{k=1}^{N-1} m_o(\xi/2^k) \right) \widehat{\varphi}(\xi/2^N).$$

Under all our assumptions of Type II, as  $N \rightarrow \infty$  the factor  $\widehat{\varphi}(\xi/2^N)$  converges to a non-zero limit. Therefore the products converge as well, to an infinite product, and we have *The Cascade Formula*:

(Cascade Formula) 
$$\widehat{\varphi}(\xi) = \widehat{\varphi}(0) \prod_{k=1}^{\infty} m_o(\xi/2^k).$$

There is a useful Theorem in our case:  $|\widehat{\varphi}(0)| = 1!$  By choosing an appropriate constant, we can (and do!) assume that  $\widehat{\varphi}(0) = 1$ . The Theorem will be stated and proved later.

First we will **review**.

We began with an MRA and introduced the *scaling equation*,

(Scaling) 
$$\varphi(t) = \sum_n h(n)\sqrt{2} \varphi(2t - n), \text{ and } \sum_n |h(n)|^2 = 1.$$

and its Fourier transform version

(Fourier Scaling) 
$$\widehat{\varphi}(\xi) = \sum_n \frac{h(n)}{\sqrt{2}} e^{-in\xi/2} \widehat{\varphi}\left(\frac{\xi}{2}\right).$$

We used (Fourier Scaling) to define the *low-pass filter*, a function  $m_o(\xi)$  whose formula is (another  $L^2$  series, this one in  $L^2(\mathbb{T})$ ):

(Low-pass Filter) 
$$m_o(\xi) := \sum_n \frac{h(n)}{\sqrt{2}} e^{-in\xi}.$$

We then used the orthonormality of  $\{\varphi(t - n) : n \in \mathbb{Z}\}$ , the integer translates of the *scaling function*  $\varphi$ , to show that the Fourier transform of the scaling function satisfies

$$\sum_n |\widehat{\varphi}(\xi + 2\pi n)|^2 = 1 \text{ a.e.,}$$

And then we showed that the low-pass filter  $m_o$  satisfies

$$|m_o(\xi)|^2 + |m_o(\xi + \pi)|^2 = 1 \text{ a.e.}$$

We used some special assumptions we called Type II assumptions to show that  $m_o$  is continuous, so it is actually true (for us) that

$$|m_o(\xi)|^2 + |m_o(\xi + \pi)|^2 \equiv 1.$$

Two of our Type II assumptions allowed us to show that  $m_o(0) = 1$ , and we went on to show that the  $h(n)$  satisfy these conditions:

(h1) 
$$\sum_n |h(n)|^2 = 1.$$

(h2) 
$$1 = m_o(0) = \sum_n \frac{h(n)}{\sqrt{2}} \text{ or, } \sum_n h(n) = \sqrt{2}.$$

$$(h3) \quad 0 = m_o(\pi) = \sum_n \frac{h(n)}{\sqrt{2}} (-1)^n \quad \text{or,} \quad \sum_n (-1)^n h(n) = 0.$$

$$(h4) \quad \sum_n h(n) \overline{h(n-2k)} = \delta_{k0} = \begin{cases} 1, & \text{if } k = 0; \\ 0, & \text{if } k \neq 0. \end{cases}$$

Using all this we found the *The Cascade Formula*:

$$(Cascade Formula) \quad \widehat{\varphi}(\xi) = \widehat{\varphi}(0) \prod_{k=1}^{\infty} m_o(\xi/2^k).$$

Here are our three Type II assumptions:

**Type II Assumption 1:**  $\sum_n |h(n)| < \infty$ .

**Type II Assumption 2:**  $\varphi \in L^1$ . That is, we suppose  $\varphi$  is an integrable function.

**Type II Assumption 3:**  $\widehat{\varphi}(0) \neq 0$ . That is, we suppose  $\int \varphi(t) dt \neq 0$ .

**end of review**

Here is a new Type II Assumption that will lead us to some very special scaling functions.

**Type II Assumption 4:**  $\varphi$  has compact support.

That is, there is a closed and bounded set, outside of which  $\varphi(t) \equiv 0$ . If we don't need to know exactly which set is the *support* of  $\varphi$ , we can just say there is a positive number  $M$  such that  $\varphi(t) \equiv 0$  if  $|t| > M$ .

We will use the new Type II assumption to show that only finitely many of the  $h(n)$  can be non-zero. Let us take another look at the Scaling Equation. All we have used so far is that, since  $V_0 \subseteq V_1$ , and since  $\varphi \in V_0$ , and  $\|\varphi\| = 1$ , we know that

$$\varphi(t) = \sum_n h(n) \sqrt{2} \varphi(2t - n), \quad \text{and that} \quad \sum_n |h(n)|^2 = 1.$$

However, we know that this representation, as a series in  $L^2$ , is unique, so by the Projection Theorem we actually know that

$$(II.4) \quad \varphi = P_{V_1} \varphi = \sum_n \langle \varphi, \sqrt{2} \varphi(2s - n) \rangle_{ds} \sqrt{2} \varphi(2t - n),$$

and so

$$h(n) = \langle \varphi, \sqrt{2} \varphi(2s - n) \rangle_{ds} = \int \varphi(s) \overline{\sqrt{2} \varphi(2s - n)} ds.$$

Now, to show that all but finitely many of the  $h(n)$  are zero, we look at the integral defining  $h(n)$ . In order that  $h(n) \neq 0$  it *has* to be true that the integrand is not identically zero. Of course, that by itself does not guarantee that  $h(n) \neq 0$ , it's just a necessary condition. We will be "crude" about the support and just say that  $\varphi(t) = 0$  if  $|t| > M$ , for some positive  $M$ . But then to have the integrand be non-zero, it has to be true that, for some  $s$ ,  $|s| \leq M$  and that (at the same time)  $|2s - n| \leq M$ . This means that, simultaneously,

$$-M \leq s \leq M \quad \text{and} \quad \frac{n - M}{2} \leq s \leq \frac{n + M}{2}.$$

Now if  $n \geq 3M$ , we'd have  $s \leq M = \frac{3M - M}{2} \leq \frac{n - M}{2} \leq s$ , which would give an integrand non-zero at at most one point and hence a zero  $h(n)$ . Thus  $n < 3M$  has to be true if  $h(n) \neq 0$ . This argument can be "multiplied by  $-1$ ." to show finally that  $h(n) = 0$  unless  $|n| < 3M$ .

Some things follow from this! Type II Assumption 1, that  $\sum_n |h(n)| < \infty$ , is automatically true if Type II Assumption 4 is true. We can also show that Type II Assumption 2, that  $\varphi \in L^1$ , is automatically true if Type II Assumption 4 is true, as follows.

With  $\varphi \in L^2$  and zero if  $|t| > M$ , the Schwarz inequality gives

$$\int |\varphi(t)| dt = \int [ |t| \leq M ] |\varphi(t)| dt \leq \left( \int [ |t| \leq M ]^2 \right)^{1/2} \left( \int |\varphi(t)|^2 \right)^{1/2} = (2M)^{1/2} \|\varphi\|_2 < \infty.$$

We can put this into the form of a Theorem.

**(II.5) Theorem:** *Suppose  $\varphi$  is in  $L^2$  and has compact support. If the conditions (i) – (v) of an MRA are true for  $\varphi$  then  $\varphi \in L^1$  (that is  $\varphi$  is integrable). If, in addition,  $\int \varphi(t) dt \neq 0$ , then the low-pass filter,  $m_o$ , is a trigonometric polynomial, i.e. it has only finitely many non-zero coefficients  $h(n)$ , so that  $\sum_n |h(n)| < \infty$ .*

There are many “useful” scaling functions that do *not* have compact support! Our main concern is to find ones that *do* have compact support. These were discovered by Ingrid Daubechies, *Orthonormal bases of compactly supported wavelets*, Communications of Pure and Applied Mathematics 41(1988)909–996.

There will be one more condition, (h5), added to the Low-pass Filter Equations. The five Low-pass Filter Conditions will lead to scaling functions and wavelets with compact support.