Please Note! *Special Problems are like "term papers." They must be well-written, on standard 8.5 x 11 paper, and must be succint - with exactly enough detail. Scoring is competitive; neatness counts.*

Paper torn from spiral notebooks is not acceptable.
Err in the direction of slightly excessive detail at first,
but prolixity is not acceptable.

Exercises 1: to accompany § I.1

A1. Let

$$u := \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},\ v := \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},\ w := \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},\ x := \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},\ y := \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix},\ z := \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

(i) Write out the *formulas* for the *linear combinations* $au + bv + cw$ and $ax + by + cz$.
Evaluate your formulas when
(ii) $a = 1,\ b = 1$ and $c = -1$;
(iii) $a = 1,\ b = -1$ and $c = 1$;
(iv) $a = 0,\ b = 1$ and $c = 2$;
(v) $a = -2,\ b = 3$ and $c = 0$;
(vi) $a = 2,\ b = -1$ and $c = -1$;
(vii) $a = 5,\ b = -1$ and $c = 0$.

A2. See A1 for the definitions of u, v and w. Find vectors p, q and r so that
(i) $u + v + p = w$,
(ii) $u - v + q = w$,
(iii) $-2u + r = w$.

A3. Using p, q and r from A2, efficiently calculate
(i) $3p + q + r$,
(ii) $p + 3q + r$,
(iii) $p + q + 3r$.

A4. Given that $x \in \mathbb{R}^n$, what does "$x = 0$" mean?

A5. Given that $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, what does "$x = y$" mean?

A6. Given that $x \in \mathbb{R}^n$, and $x \neq 0$, verify that the *set* L_x of *all* vectors tx, where $t \in \mathbb{R}$ (i.e., all *scalar multiples* of x), satisfies **V1** – **V8.** We call L_x the *span* of $\{x\}$, and write $\operatorname{span}\{x\}$ for short.

A7. Given that $x \in \mathbb{R}^n$, and $y \in \mathbb{R}^n$, verify that the *set* $L_{x,y}$ of *all* vectors $ax+by$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$ (i.e., all *linear combinations* of x and y), satisfies **V1** – **V8.** We call $L_{x,y}$ the *span* of $\{x, y\}$, and write $\operatorname{span}\{x, y\}$ for short.

A8. Given a set S contained in a vector space V, we write $\operatorname{span} S$ to denote the set of all linear combinations of finitely many vectors in S. *Linear combinations are always finite sums, even when the set S is an infinite set!!!* In $\mathbb{R}^n$, what is the span of the empty set? Why? What is the span of $\mathbb{R}^n$? Why?

A9. In A7, verify that, if $y \in \operatorname{span}\{x\}$, then $\operatorname{span}\{x, y\} = \operatorname{span}\{x\}$.

A10. In A7, verify that, if $y \notin \operatorname{span}\{x\}$, then $\operatorname{span}\{x, y\} \neq \operatorname{span}\{x\}$ and $\operatorname{span}\{x, y\} \neq \operatorname{span}\{y\}$.

Exercises 2: to accompany § I.2

A1. Verify that if $S \subseteq \mathbb{R}^n$, then $\operatorname{span} S$ is a subspace of $\mathbb{R}^n$.

A2. Given that $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ and $x \neq y$, the *set* $L(x, y)$ of all vectors $z(t) := (1 - t)x + ty$ is called the *line determined by* x and y, or the *line thru* x and y. We often think of $z(t)$ as the location of a particle **p** in $\mathbb{R}^n$ at time t. Note where **p** is when $t = 0$ and when $t = 1$. Suppose that **p** is located at the origin, 0, at some time T. That is, $z(T) = 0$. Verify that in this case, $L(x, y)$ is a subspace of $\mathbb{R}^n$. On the other hand, verify that if $\mathbf{p}$ is never 0, then $L(x, y)$ is *not* a subspace of $\mathbb{R}^n$.

A3. Find *all* solutions of the following sets of simultaneous equations (they are *linear* and *homogeneous* equations):

$$\begin{cases} x+y=0 \\ x+z=0 \\ y+z=0 \end{cases}, \quad \begin{cases} x+2y+3z=0 \\ 4x+5y+6z=0 \\ 7x+8y+9z=0 \end{cases}, \quad \begin{cases} x+3y=0 \\ x+2y=0 \end{cases}, \quad \begin{cases} x+3y+5z=0 \\ 2x+4y+6z=0 \end{cases}.$$

A4. Find a system of 3 linear equations in 3 unknowns that has $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ as its unique solution.

A5. Find a system of 3 linear equations in 3 unknowns that has $t\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ as its solutions, where all $t \in \mathbb{R}$ appear.

A6. Find *all* solutions of the following sets of simultaneous equations (they are *linear* and *inhomogeneous* equations):

$$\begin{cases} x+y=0 \\ x+z=1 \\ y+z=1 \end{cases}, \quad \begin{cases} x+2y+3z=6 \\ 4x+5y+6z=15 \\ 7x+8y+9z=24 \end{cases}, \quad \begin{cases} x+3y=1 \\ x+2y=0 \end{cases}, \quad \begin{cases} x+3y+5z=7 \\ 2x+4y+6z=8 \end{cases}.$$

A7. Test for linear independence:

(i) $$S_1 := \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$

(ii) $$S_2 := \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\},$$

(iii) $$S_3 := \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\},$$

(iv) $$S_4 := \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\},$$

(v) $$S_5 := \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\},$$

(vi) $$S_6 := \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \end{pmatrix} \right\},$$

(vii) $$S_7 := \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 7 \\ 11 \end{pmatrix}, \begin{pmatrix} 13 \\ 17 \\ 19 \end{pmatrix}, \begin{pmatrix} 23 \\ 29 \\ 31 \end{pmatrix} \right\}.$$

A8. Given that a, b and c are non-zero scalars, test for linear independence:

(i) $\{a\mathbf{i}+b\mathbf{j}+c\mathbf{k},\ a\mathbf{i}-c\mathbf{j}+b\mathbf{k}\}$, (ii) $\{a\mathbf{i}+b\mathbf{j}+c\mathbf{k},\ -c\mathbf{j}+b\mathbf{k},\ b\mathbf{i}-a\mathbf{j}\}$, (iii) $\{a\mathbf{i}+b\mathbf{j},\ a\mathbf{i}+c\mathbf{k},\ b\mathbf{j}+c\mathbf{k}\}$.

A9. Test these sets of polynomials for linear independence:

$$\{1+x,\ x+x^2\},\ \{1,\ x,\ x^2,\ 3+2x+x^2\},\ \{1,\ 1+x,\ 1+x+x^2,\ 3+2x+x^2-x^3\},\ \{1+x^2,\ 1-x^2\}.$$

A10. Verify that $\{\sin x, \cos x\}$, $\{\sin x, \sin 2x, \sin 3x\}$, and $\{1, e^x, e^{-x}\}$ are linearly independent sets of functions on $\mathbb{R}$.

A11. In 1.A1, which sets of three of the vectors u, v, w, x, y, z are linearly independent?

Exercises 3: to accompany § I.3

A1. In 1.A1, express each of x, y and z as a linear combination of u, v and w.

A2. Show that, in 1.A1, $\{u, v, w\}$ is *not* an *orthogonal* set, and that $\{x, y, z\}$ *is* an *orthogonal* set, but is not an *orthonormal* set.

A3. In 1.A1, express each of u, v and w as a linear combination of x, y and z.

A4. If $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, we define their *dot product* to be the number

$$x \bullet y := \sum_{i=1}^{n} x_i y_i \text{ and we define the } \textit{length} \text{ of } x,\ |x|, \text{ by } |x|^2 := x \bullet x.$$

Verify that the dot product is an inner product on $\mathbb{R}^n$, namely that it satisfies **SP1 − SP3.**

A5. In $\mathbb{R}^n$, the span of a linearly independent set consisting of of two vectors has all the properties of a plane. Thus verify, using the Law of Cosines, if you wish, that $x \bullet y = |x||y| \cos\theta$, where θ denotes the angle between vectors x and y, usually measured in radians, which we take to satisfy $0 \le \theta \le \pi$.

A6. Find a vector formula for the line thru 0 and the average of the vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. Find two points $\mathbf{p}_+$ and $\mathbf{p}_-$ on this line so that the distance from each to $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ is equal to the distance between distinct pairs of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. The points $\mathbf{p}_+$ and $\mathbf{p}_-$ should correspond to points with positive and negative values of t, respectively, on the line thru 0 and the average of the vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. Plot the points $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, $\mathbf{p}_+$ and $\mathbf{p}_-$. Draw the segments joining each pair from $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and $\mathbf{p}_+$. What figure do you get? What figure do you get if you now draw the segments from $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ to $\mathbf{p}_-$)? Your first figure is a regular polyhedron, but the second is not. Why not?

A7. Given m vectors, $x_1, \ldots, x_m$ in $\mathbb{R}^n$ such that $|x_i - x_j| = 1$ if $i \ne j$. Why must $m \le n+1$?

A8. Apply the Gram-Schmidt process to the set $\{u, v, w\}$ of vectors in 1.A1, to obtain an orthonormal set.

A9. Verify that for any two vectors x and y in an inner product space, $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$. Why is this called the "Parallelogram Law?"

A10. The Polarization Identity allows us to compute $\langle x, y\rangle$ in terms of $\|x+y\|^2$ and $\|x-y\|^2$:

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right). \text{ Verify this.}$$

A11. Suppose that $f(x)$ and $g(x)$ are continuous functions on $[0, 1]$. Suppose that $f(x) + g(x) = 1 + x^2$ for every x in $[0, 1]$ and that $f(x) - g(x) = 2x$ for every x in $[0, 1]$. Calculate $\int_0^1 f(x)g(x)\,dx$.

A12. How can we calculate the angle between two planes in $\mathbb{R}^3$? Between two hyperplanes in $\mathbb{R}^n$?

A13. Find an equation for the plane that contains the points $\mathbf{i}+\mathbf{j}$, $\mathbf{i}+\mathbf{k}$ and $\mathbf{j}+\mathbf{k}$.

A14. Find a vector formula for the line thru the origin that is perpendicular to the plane in A13.

A15. In $\mathbb{R}^3$ there is a *vector* product as well as a scalar product. However, the product of x and y (in that order) is not equal to the product of y and x. We begin by defining

$$\mathbf{i}\times\mathbf{j} = \mathbf{k}, \quad \mathbf{j}\times\mathbf{k} = \mathbf{i}, \quad \mathbf{k}\times\mathbf{i} = \mathbf{j}, \quad \text{while } \mathbf{j}\times\mathbf{i} = -\mathbf{k}, \quad \mathbf{k}\times\mathbf{j} = -\mathbf{i}, \quad \text{and } \mathbf{i}\times\mathbf{k} = -\mathbf{j}.$$

We then define $x \times y$ by writing each of them as linear combinations of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, and expand the product as usual, converting the products of the basis vectors to basis vectors or their negatives according to the "cyclic order" rules above. Verify that $x \times y = -y \times x$. What does that tell us about $x \times x$? This product is called the "cross product" of x and y. Is it true, or not, that we always have $x \times (y \times z) = (x \times y) \times z$?

A16. Show that for every pair $x,\ y$ in $\mathbb{R}^3$, $x \times y \perp x$ and $x \times y \perp y$.

A17. Given: x and y, both in $\mathbb{R}^n$, are non-zero and are not proportional. Find the value of t such that $z(t) := (1-t)x + ty$ is closest to the origin. Your answer could involve x and $y - x$. Find the closest distance, perhaps in terms of the angle between x and $y - x$.

A18. Consider the plane whose equation is

$$ax + by + cz = d, \quad \text{where} \quad a^2 + y^2 + z^2 > 0 \quad \text{and} \quad d \neq 0.$$

Find the point on this plane that is closest to the origin.

A19. Find the limit, as $p \to \infty$, of $(|x|^p + |y|^p + |z|^p)^{\frac{1}{p}}$ When $x = -1$, $y = 2$ and $z = 3$. Replace x by -4 and repeat. What about the general case? The limit is denoted $\|\mathbf{v}\|_\infty$, where $\mathbf{v} = (x,\ y,\ z)$.

Exercises 4: to accompany § I.4

A1. A function $f : \mathbb{R}^2 \to \mathbb{R}^3$ is given to be linear.

It is also given that $f(\begin{pmatrix} 1 \\ 1 \end{pmatrix}) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and that $f(\begin{pmatrix} 1 \\ -1 \end{pmatrix}) = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$. Find the matrix for f.

A2. Find the matrix for the linear function $f : \mathbb{R}^2 \to \mathbb{R}^2$ that reflects a vector in the line $x = y$.

A3. Find the equation in polar coordinates for the circle whose equation in plane rectangular coordinates is $(x-1)^2 + y^2 = 1$.

A4. What is the rectangular-coordinate equation corresponding to the polar-coordinate equation $r = \tan\theta \sec\theta$? This is tricky to handle!

A5. The *transpose* of an $m \times n$ matrix $A = (a_{ij})$ is denoted by A^T and it has entries given by $A^T = (a_{ji})$; in words, the rows of A become the columns of A^T, so A^T is an an $n \times m$ matrix. What sort of linear mapping is the transpose of the 2×2 matrix giving a clockwise rotation thru the angle θ?

A6. Given vectors $\mathbf{v}$ in $\mathbb{R}^m$ and $\mathbf{w}$ in $\mathbb{R}^n$, does the matrix product $\mathbf{v}\mathbf{w}^T$ make sense? What is it?

A7. Verify that if A is a matrix, then the ith row of A is $\mathbf{e}_i^T A$, and the jth column of A is $A\mathbf{e}_j$, where the vectors $\mathbf{e}_k$ are the standard basis vectors with zeros in each row except the ith, which has a 1 in it. Note that the $\mathbf{e}_k$ depend on context!

A8. The "usual" way to multiply matrices A and B tells us to compute the ijth entry of AB by dotting the ith row of A and the jth column of B. Write this dot product as an actual matrix product, as in A7, when $A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 \end{pmatrix}$ and $B = A^T$ and for each possible $i = j$. Do the same when the order of the factors is reversed.

A9. Verify that if A has n columns and B has n columns, then $AB = \sum_{j=1}^{n} A\mathbf{e}_j\mathbf{e}_j^T B$. Suggestion: Look for the standard formula $\sum_{k=1}^{n} a_{ik}b_{kj}$ to show up.

A10. Given that $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 7 & 9 \end{pmatrix}$ and that $B = -\frac{1}{3}\begin{pmatrix} 1 & 1 & -1 \\ 4 & -5 & 2 \\ -4 & 3 & -1 \end{pmatrix}$, calculate AB and BA.

A11. Solve

$$\begin{aligned} x + 2y + 3z &= 1 \\ 4x + 5y + 6z &= 2 \\ 8x + 7y + 9z &= 3. \end{aligned}$$

Take advantage of A10! Repeat when the 2 and 3 on the right are intercahnged.