

**Note:** PP(xx) refers to the reference number (xx) in “The Peano Postulates, version 4,” and ES(xx) refers to the reference number (xx) in “The Exponential Saga.”

### Exercises 15

Proposed for Apr 30

Exercises # 1 – # 7 offer a way to prove that for every positive  $y$  there exists a unique real number  $x$  such that  $e^x = y$ . We then say that  $x$  is the *natural logarithm* of  $y$ . This way avoids the use of the Intermediate Value Theorem.

- Let  $y$  be a positive real number. Prove that there exists one and only one  $k \in \mathbb{Z}$  such that  $2^k \leq y < 2^{k+1}$ .
- Let  $y$  be a positive real number. Prove that there exists one and only one  $k \in \mathbb{Z}$  such that  $e^k \leq y < e^{k+1}$ . There is a geometric idea here; the set of positive numbers is the union of the disjoint intervals  $[e^m, e^{m+1})$  as  $m$  runs thru *all* the integers. We proved things in ES that let us how that  $\{e^m\}$  increases strictly from 0 to  $+\infty$  as  $m$  increases in  $\mathbb{Z}$  from  $-\infty$  to  $+\infty$ .
- For a given  $y > 0$  let us denote the  $k$  found in # 2 by  $k_0$ . Then we apply the method in # 2 to  $y^2$ , and let  $k_1$  denote the  $k$  we get for  $y^2$ . We continue this process, letting  $k_n$  denote the  $k$  we get for  $y^{2^n}$  by applying the method in # 2 to  $y^{2^n}$ . What sequence  $\{k_n\}$  do we get if  $y = 1$ ? If  $y = e$ ? If  $y = e^\ell$  for some integer  $\ell$ ? What are the first five terms of the sequence  $\{k_n\}$  when  $y = 2$ ? Use a calculator!
- Prove that if  $y > 1$  the sequence  $\{k_n\}$  found in # 3 is an increasing sequence. It might be useful to notice that  $[e^m, e^{m+2}) = [e^m, e^{m+1}) \cup [e^{m+1}, e^{m+2})$ , and so on.
- Prove that  $e^{k_n/2^n} \rightarrow y$  as  $n \rightarrow \infty$  for each  $y > 0$ .
- As an alternative way to solve # 4, show that  $2k_n \leq k_{n+1} \leq 2k_n + 1$ . One clever way to do this is to notice that  $k_n + 1$  is the smallest integer  $m$  such that  $e^m > y^{2^n}$ , and therefore  $k_n$  is the largest integer  $m$  such that  $e^m \leq y^{2^n}$ . This uses the Well Ordering Theorem. The inequality  $k_{n+1} \leq 2k_n + 1$  is easier to prove than  $2k_n \leq k_{n+1}$ ; the latter seems to require looking at “cases.”
- Prove that  $\{k_n/2^n\}$  converges to a number  $x$ . Use the continuity of  $e^t$ , proved in ES(5.1) – ES(5.3)+, to show that  $e^x = y$ . Use the strict increase of  $e^t$  to show that only one  $x$  can satisfy  $e^x = y$ .
- If we write  $\log y$  to denote  $x$ , use the Chain Rule and the equation  $e^{\log y} = y$  to find the derivative of  $\log y$  with respect to  $y$ , *assuming* that  $\log y$  is differentiable.
- Show that  $\log e^x = x$  for all real  $x$ .
- Show that  $\log y_1 y_2 = \log y_1 + \log y_2$  for all positive numbers  $y_1$  and  $y_2$ . What is  $\log 1$ ?  $\log e$ ?  $\log(1/y)$ , in terms of  $\log y$ ?

### Exercises 14

Proposed for Apr 18

- Suppose that  $\sum z_n$  is a series with complex terms,  $z_n = x_n + iy_n$ , where  $x_n$  and  $y_n$  are real numbers. Prove that  $\sum z_n$  converges if and only if  $\sum x_n$  and  $\sum y_n$  converge.

2. **Telescoping Series** Find  $s_N$  for each series. If the series converges, find its sum.

If the series diverges, say why.

(a)  $\sum_{n=3}^{\infty} \frac{1}{n^2 - 4}$  (b)  $\sum_{n=1}^{\infty} \frac{n}{(n+2)(n+3)}$  (c)  $\sum_{n=1}^{\infty} \frac{1}{n^2 + 5n + 4}$  (d)  $\sum_{n=1}^{\infty} \frac{1}{n^2 + (m+1)n + m}$ , where  $1 < m \in \mathbb{Z}^+$ .

3. Test the following series for convergence. Justify your answers! If  $x$  appears in the series,  $x > 0$  is assumed.

(a)  $\sum_{n=2}^{\infty} \binom{n}{2} x^n$ ; (b)  $\sum_{n=m}^{\infty} \binom{n}{m} x^n$ ,  $m \in \mathbb{Z}^+$ ; (c)  $\sum_{n=1}^{\infty} \frac{x^n}{n}$ ; (d)  $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ ; (e)  $\sum_{n=1}^{\infty} \frac{n^2 x^n}{2^n}$ ; (f)  $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ ; (g)  $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$ .

4. Test the following series for convergence. Justify your answers!

(a)  $\sum_{n=3}^{\infty} \frac{1}{\sqrt{n^2 - 4}}$  (b)  $\sum_{n=3}^{\infty} \frac{1}{\sqrt{n^3 - 8}}$  (c)  $\sum_{n=3}^{\infty} \frac{n}{\sqrt{n^3 - 8}}$  (d)  $\sum_{n=3}^{\infty} \sqrt{\frac{n}{n^3 - 8}}$  (e)  $\sum_{n=3}^{\infty} \sqrt{\frac{n}{n^4 - 16}}$  (f)  $\sum_{n=3}^{\infty} \sqrt{\frac{n^2}{n^5 - 32}}$

5. Let  $x > 0$ . Test the following series for convergence. Justify your answers! Use # 4 for part of each answer.

(a)  $\sum_{n=3}^{\infty} \frac{x^n}{\sqrt{n^2 - 4}}$  (b)  $\sum_{n=3}^{\infty} \frac{x^n}{\sqrt{n^3 - 8}}$  (c)  $\sum_{n=3}^{\infty} \frac{nx^n}{\sqrt{n^3 - 8}}$  (d)  $\sum_{n=3}^{\infty} \sqrt{\frac{nx^n}{n^3 - 8}}$  (e)  $\sum_{n=3}^{\infty} \sqrt{\frac{nx^n}{n^4 - 16}}$  (f)  $\sum_{n=3}^{\infty} \sqrt{\frac{n^2 x^n}{n^5 - 32}}$

6. Test  $\sum_{n=1}^{\infty} a_n$  for convergence where, in each case,

(a)  $a_{2n} = 1/4^n$ ,  $a_{2n+1} = 1/5^n$ ; (b)  $a_n = \frac{1}{kn}$ , where  $2^{k-1} \leq n < 2^k$  so the first few  $a_n$  are  $1, \frac{1}{4}, \frac{1}{6}, \frac{1}{12}, \frac{1}{15}$ .

7. Test for convergence. Justify your answers! Here,  $x$  is arbitrary.

(a)  $\sum_{n=0}^{\infty} (-1)^n$  (b)  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$  (c)  $\sum_{n=0}^{\infty} x^n$  (d)  $\sum_{n=1}^{\infty} nx^n$  (e)  $\sum_{n=1}^{\infty} \frac{x^n}{n}$  (f)  $\sum_{n=1}^{\infty} e^n \frac{x^n}{\sqrt{n}}$  (g)  $\sum_{n=1}^{\infty} \frac{x^{n^2}}{n}$

8. Here the coefficients  $a_n$  are the ones in # 6 and  $x$  is arbitrary. Test each  $\sum a_n x^n$  for convergence.

9. Prove that if  $\sum z_n$  and  $\sum w_n$  are convergent series with complex terms then

(a)  $\sum(z_n + w_n)$  converges; (b)  $\sum(z_n - w_n)$  converges. (c) Must  $\sum z_n w_n$  converge?

10. Suppose that  $\{s_n\}$  converges. Prove that the series  $\sum_{n=1}^{\infty} (s_{n+1} - s_n)$  converges.

### Exercises 13

Proposed for Apr 9

1. For  $m = 2, 3, 4, 5, 6$  find a corresponding  $N = N_m$  such that  $\left(1 + \frac{x}{N}\right)^N > \left(1 - \frac{1}{m}\right) \sum_{r=0}^m \frac{x^r}{r!}$ . for all  $x > 0$ .

2. Differentiate the series  $E(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$  term-by-term. What do you get?

Do the same for the series  $L(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$ . What do you get?

It would thus be useful to know that certain series, when differentiated term-by-term, give a series that is indeed the derivative of the function represented by the original series! This works for  $E(x)$  for all  $x$ , but works for  $L(1-x)$  only for  $|x| < 1$ .

3. Prove that, if  $|x| < 1$ , then  $|e^x - 1 - x| < ex^2/2$ . Suggestions: Use  $f_n(x)$  with  $n > 2$ . Use the Binomial

Theorem. Prove that  $\frac{n(n-1)(n-2)^k}{(k+2)!n^{k+2}} < 1/2$  for  $0 \leq k \leq n-2$  so that you can transform the absolute value of the difference into  $(x^2/2)f_{n-2}(|x|)$ . This gives an alternate way to prove that the derivative of  $e^x$  at  $x=0$  exists and has value 1.

4. Suppose that  $y_1 > 0$  and  $y_2 > 0$ , and that  $y_1 = e^{x_1}$  and  $y_2 = e^{x_2}$ . Are  $x_1$  and  $x_2$  unique? Find  $x_3$  and  $x_4$  so that  $y_1 y_2 = e^{x_3}$  and  $y_1/y_2 = e^{x_4}$ . What are  $e^{x_3+x_4}$  and  $e^{x_3-x_4}$ ?

5. Suppose that  $y > 0$  and that  $y = e^x$ . Find the  $n$ -th root of  $y$  in terms of  $x$ .

6. Given:  $e^x$  is continuous,  $e^0 = 1$  and  $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$ . Find (and prove, using what was given)

$$\lim_{x \rightarrow 0} \frac{e^{2x} - e^x}{x} \left( = \lim_{x \rightarrow 0} \frac{e^{2x} - e^x}{2x - x} \right) \text{ and } \lim_{x \rightarrow 0} \frac{e^{x^2} - e^x}{x^2 - x}.$$

7. Using that  $(e^x)' = e^x$  and Calculus, prove that for  $x$  positive and  $n$  a positive integer,  $g_n(x) := x^n e^{-x}$  increases at first as  $x$  increases from 0, then decreases. Find the point  $x_n$  at which the change from increasing to decreasing occurs, and find the limit, as  $x \rightarrow \infty$ , of  $g_n(x)$ . Do not use l'Hospital's rule unless you prove it. Hint: you can get a useful upper bound for  $g_n(x)$  by replacing  $e^x$  by  $f_{n+1}(x)$ . The Binomial Theorem is also handy here, along with the Squeeze Theorem.

8. Suppose you deposit  $\$A$  in a savings account that has an interest rate of  $x\%$ . At the end of  $k$  years, what is your balance if the interest is compounded: (a) semi-annually, (b) quarterly, (c) annually. Express your answers in terms of  $A$ , the  $f_n$ ,  $x$  and  $k$ . (d) What would "continuous compounding" mean, and what would the answer be then?

9. Suppose you have a moderately large balance of  $\$B$  on a credit card that charges 1% per month interest, and requires that you pay 10% of the balance each month. You make no further charges and pay 10% of the balance each month. What will your balance be just after your twelfth payment? How much interest will you pay? What is your effective annual interest rate? How long will it take until the balance is less than  $B/10$ ?

10. Prove that, if  $x \neq y$  are real numbers, then  $e^{(x+y)/2} < \frac{e^x + e^y}{2}$ .

### Exercises 12

Proposed for Mar 28

1. Suppose that  $n$  is an arbitrary positive integer such that  $n > 2$ .

Prove that, for  $0 \leq k \leq (n-2)/2$ , it is true that  $\binom{n}{k} < \binom{n}{k+1}$ .

- Prove that  $\left\{ \left(1 + \frac{1}{n}\right)^{n+1} \right\}$  is a decreasing sequence.
- Calculate  $\sum_{k=0}^n \binom{n}{k} \left(-\frac{1}{2}\right)^k$ ,  $\sum_{k=0}^n \binom{n}{k}$  and  $\sum_{k=0}^n \binom{n}{k} (-2)^k$ .
- Prove that the series  $E(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!}$   $\left( = \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{x^k}{k!}, \text{ by definition!} \right)$  converges for each  $x > 0$ . This is easy if you use what was done in class!
- Given that  $x > 1$ , prove that there is a positive integer  $K$  such that  $\frac{x^k}{k!}$  increases for  $0 \leq k \leq K$ , then decreases strictly for  $k > K$ . Find  $K$  for  $x = 2, 4$  and  $8$ . Find  $\lim_{k \rightarrow \infty} \frac{x^k}{k!}$ . Justify your answer!
- Find  $\lim_{n \rightarrow \infty} \left(1 - \frac{m}{n}\right)^m$ . Prove that, for each positive integer  $m$  and each  $\epsilon$  with  $0 < \epsilon < 1$ , there exists a positive integer  $n$  so large that  $\frac{n!}{(n-r)!n^r} > (1 - \epsilon)$  for  $0 \leq r \leq m$ .
- Use the result in # 6 and the methods used in class to show that for each positive  $x$ , each  $\epsilon$  with  $0 < \epsilon < 1$ , and each positive integer  $m$ , there exists a positive integer  $n$  so large that

$$\left(1 + \frac{x}{n}\right)^n > (1 - \epsilon) \sum_{r=0}^m \frac{x^r}{r!}.$$

- With  $f(x) := \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n$ , and  $E(x)$  defined in # 4, use # 7 to show that  $f(x) \geq E(x)$  for all  $x > 0$ .
- In class, we showed that for  $x > 0$ ,

$$\left(1 + \frac{x}{n}\right)^n < \sum_{r=0}^n \frac{x^r}{r!}.$$

Use this and # 8 to prove that  $f(x) = E(x)$  for all  $x \geq 0$ .

- Calculate  $\sum_{k=0}^{\infty} \frac{2^k}{3^{k+1}}$ ,  $\sum_{k=0}^{\infty} \frac{3^k}{4^{k+1}}$  and, for  $m \in \mathbb{Z}^+$ ,  $\sum_{k=0}^{\infty} \frac{m^k}{(m+1)^{k+1}}$ .

### Exercises 11

Proposed for Mar 14

- Suppose  $z = x + iy \neq 0$ . Find the real and imaginary parts of  $1/z$ .
- Suppose  $z = x + iy$ , where  $x > y > 0$ . Plot  $z$  and  $z^2$  in the plane, as line segments from 0 to the points  $z$  and  $z^2$ . Use analytic geometry methods or trigonometry to show that the angle between  $z^2$  and the positive  $x$ -axis is twice the angle between  $z$  and the positive  $x$ -axis.
- Plot the points that are in the sequence  $\{i^n\}$ .
- Find all *complex* solutions  $z$  of the equation  $z^2 = i$ . Hint: use Exercise 2.
- Find all *complex* solutions  $z$  of the equation  $z^3 = 1$ . Hint: use Exercise 2.
- Suppose  $z = x + iy \neq 0$ . Find all *complex* solutions  $w$  of the equation  $w^2 = z$ . Suggestions: write  $w = u + iv$  and set up the equation  $w^2 = z$  as two equations with unknowns  $u$  and  $v$ . Consider three cases:  $y = 0$ ,  $y > 0$  and  $y < 0$ . Use the quadratic formula. Simplify the formula for  $v$  when  $y \neq 0$  by using “rationalization:”  $\frac{1}{a+b} = \frac{1}{a+b} \frac{a-b}{a-b}$ , where  $a+b \neq 0 \neq a-b$ , to get rid of a complicated square root in the denominator.
- Find all *complex* solutions  $z$  of the equation  $z^4 = 1$ .
- Plot all *complex* solutions  $z$  of the equation  $z^5 = 1$ . Don’t seek formulas for them. Explain your method.
- Are the Difference-of-Powers Formula and the Binomial Theorem true for complex numbers? Explain why.
- Prove that multiplication of complex numbers is commutative and associative.

### Exercises 10

Proposed for Mar 7

- Prove that if  $x_n \rightarrow L$  and  $x_n \rightarrow M$  then  $M = L$ . This is called “uniqueness of limits.”
- Prove that  $\{(-1)^n\}$  diverges.

Exercises 3 – 10 form a related set.

3. Let us define a sequence, recursively, as follows:

$$x_1 = 2 \quad \text{and, for } n \geq 1, \quad x_{n+1} := \frac{1}{2} \left( \frac{2}{x_n} + x_n \right).$$

Calculate (use your calculator if you wish)  $x_n$  for  $2 \leq n \leq 5$ , and plot your points  $(x_n, x_{n+1})$ . What function  $f(x)$ 's graph do your points lie on?

4. Calculate  $x_{n+1}^2 - 2$ , in terms of  $x_n$ , and write your answer as the square of something.

5. Calculate  $x_{n+1}^2 - 2$  in terms of  $x_n^2 - 2$  and  $x_n^2$ , and so prove that for all  $n$ ,  $x_n^2 > 2$ .

6. Calculate  $x_{n+1} - x_{n+2}$  in terms of  $x_n^2 - 2$  and  $x_{n+1}$  and so prove that the sequence  $\{x_n\}$  is decreasing (the first few inequalities you'll be able to see from # 3).

7. Given  $\epsilon > 0$  find  $\delta > 0$  such that for a given  $x > 0$ , if  $y > 0$  and  $|x - y| < \delta$ , then  $\left| \left( \frac{2}{x} + x \right) - \left( \frac{2}{y} + y \right) \right| < \epsilon$ .

Your  $\delta$  will depend on  $x$  as well as  $\epsilon$ . You are being asked to find a formula for  $\delta$ , in terms of  $\epsilon$  and  $x$ . Use the triangle inequality,  $|a + b| \leq |a| + |b|$ , without proof.

8. Prove that the sequence from # 3 is bounded below.

9. Exercises # 6 and # 9 together imply that  $\{x_n\}$  (as defined in # 3) converges. Use # 7 to find out what the limit is.

10. What happens if, in # 3,  $x_{n+1} := \frac{1}{2} \left( \frac{3}{x_n} + x_n \right)$ ?

### Exercises 9

Proposed for Mar 5

1. Prove that for positive  $m$  and  $n$ , and all real  $x$ , that  $(x^m)^n = x^{mn}$ . Do the same if they are simply integers and  $x \neq 0$ . The rule  $x^{-n} = 1/x^n$  when  $x \neq 0$  is one you brought with you. It is actually a definition!

2. Prove that if  $m \in \mathbb{Z}$  and  $n \in \mathbb{Z}$  and  $x \neq 0$  then  $(x^m)^n = (x^n)^m$ .

3. Let  $r$  be a rational number. That is,  $r = m/n$ , where  $m \in \mathbb{Z}$  and  $n \in \mathbb{Z}^+$ . We would like to **define**  $x^r$  for  $x > 0$  by  $x^r := \sqrt[n]{x^m}$ . For convenience, let us write  $y^{1/n}$  for  $\sqrt[n]{y}$ . But maybe we should define it by  $(x^{1/n})^m (= (\sqrt[n]{x})^m)$ ! To show that either way works, prove that  $(x^m)^{1/n} = (x^{1/n})^m$ . Supply appropriate quantifiers!

4. What if one person uses  $r = m/n$  and another uses  $r = k/\ell$ , where  $m$  and  $k$  belong to  $\mathbb{Z}$ , and  $n$  and  $\ell$  are positive integers? For example,  $1/2 = 3/6$ . State and prove an appropriate Theorem.

5. Exercises #3 and #4 allow us to unambiguously define  $x^r$  when  $r$  is rational, for  $x > 0$ . We had to use the usual rules for **integer** exponents that you "brought with you" especially to handle **negative** rational exponents. Write down the usual rules for integer exponents.

6. Prove that if  $r$  is a positive rational number and  $0 < x < y$  then  $x^r < y^r$ . State and prove the corresponding Theorem for negative rational exponents. It will be a corollary of the first result.

7. Prove that if  $x > 1$  and if  $r$  and  $s$  are rational numbers with  $s < r$  then  $x^s < x^r$ . State and prove the corresponding result for  $0 < x < 1$ . What happens if  $x = 1$ ?

8. Prove that, if  $x > 0$  and  $r$  and  $s$  are rational then  $(x^r)^s = x^{rs}$ .

9. Write a more detailed proof of Theorem 2.1.3.4, given that  $\alpha$  is a positive rational number, using earlier exercises in this set, perhaps.

10. Carry out the details of proving 2.1.4.3 for *rational*  $\alpha > 0$ .

### Exercises 8

Proposed for Feb 26

1. Prove that if  $|x| < 1$  then  $\{n^2 x^n\}$  is a null sequence.

2. Prove that if  $|x| \geq 1$  then  $\{n^3 x^n\}$  is **not** a null sequence.

3. Prove that if  $|x| < 1$  then  $\{n^3 x^n\}$  is a null sequence.

4. Prove that, if  $m \in \mathbb{N}$  and  $n \in \mathbb{N}$  and  $m \leq n/2$  then  $\binom{n}{m} > \frac{n^m}{2^m m!}$ .

5. Prove that if  $|x| < 1$  and  $m \in \mathbb{Z}^+$  then  $\{n^m x^n\}$  is a null sequence.

6. Give two proofs that if  $0 < s < t$  then  $s^n < t^n$  for all  $n \in \mathbb{Z}^+$ , one by induction and one using the Difference-of-Powers Formula (DoPF).

7. Use the DoPF to prove that, if  $s > 0$  and  $t > 0$  then for all  $n \in \mathbb{Z}^+$ ,  $s < t \iff s^n < t^n$ .

8. Prove that  $\{\frac{n+1}{n}\}$  is a strictly decreasing sequence, and that  $\{\frac{n}{n+1}\}$  is a strictly increasing sequence.

**Note:** To prove that a sequence  $\{x_n\}$  of positive numbers is strictly increasing, there are two main methods:

(1) prove that  $x_n/x_{n+1} < 1$ ; (2) prove that  $x_{n+1} - x_n > 0$ . Method (2) works even if the  $x_n$ 's are not all positive.

9. Prove that if  $x > 1$  then  $\{x^n\}$  is a strictly increasing sequence.

10. Prove that if  $x > 1$  then  $\{x^n/n\}$  is **eventually** a strictly increasing sequence.

### Exercises 7

Proposed for Feb 14

We keep in mind that  $\{1/n\}$  is a null sequence.

1. Prove that  $\{1/n^2\}$  is a null sequence.

2. Prove that  $\{1/2^n\}$  is a null sequence.

3. Prove Theorem 2.1.3.2 (the second Theorem in a list on page 21). Make sure your proof is complete.

4. Prove that for all natural numbers  $n$ ,  $2^n > n$ .

5. Find the first natural number  $N$  such that for all natural numbers  $n \geq N$ ,  $2^n > n^2$ .

6. Find the first natural number  $N$  such that for all natural numbers  $n \geq N$ ,  $2^n > n^3$ .

7. Prove that  $\{n^2/2^n\}$  is a null sequence. Hints:  $n^2 = n^3/n$ ; # 6, # 3.

8. Prove that, if  $c \in \mathbb{R}$  and  $\{x_n\}$  is a null sequence, then  $\{cx_n\}$  and  $\{x_n^2\}$  are null sequences.

9. Prove that for all real numbers  $a$  and  $b$ ,  $|ab| \leq \frac{a^2 + b^2}{2}$ .

10. Prove that, if  $\{x_n\}$  and  $\{y_n\}$  are null sequences, so are  $\{x_n + y_n\}$  and  $\{x_n y_n\}$ .

### Exercises 6

Proposed for Feb 12

1. Prove that Axiom (3) for the Real Numbers can be replaced by

(3') there exists  $z \in \mathbb{R}$  such that for all  $x \in \mathbb{R}$ ,  $x + z = x$ . What you have to do is to prove that there is only one element of  $\mathbb{R}$  that acts like zero.

2. Formulate an Axiom (7') to replace (7), analogously to what was done in #1.

3. Prove that your Axiom (7') "replaces" (7).

4. Since, for real numbers  $x$  and natural numbers  $n$ ,  $x^n$  is not defined in the axioms, we have to define it ourselves. Here is the definition (made using induction): For all  $x \in \mathbb{R}$ ,  $x^0 := 1$ . For all  $x \in \mathbb{R}$ , and all  $n \in \mathbb{N}$ , if  $x^n$  is defined, then  $x^{n+1} := x \cdot x^n$ . We can verify that  $x^n$  is defined for all  $n \in \mathbb{N}$ . This defines a sequence  $y_n := x^n$ . Prove that for all  $x \in \mathbb{R}$  and for all  $m$  and  $n$  in  $\mathbb{N}$ ,  $x^{m+n} = x^m x^n$  (that is,  $y_{m+n} = y_m y_n$ ).

5. From now on, let's write  $\mathbb{Z}^+$  for the positive integers, instead of  $\mathbb{P}$ . That is,  $\mathbb{Z}^+ := \mathbb{P}$ . Let  $x_n := 1/n$ , for  $n \in \mathbb{Z}^+$ , so that  $\{x_n\}$  is a sequence that starts with index  $n = 1$ . Use the Theorem of Archimedes to prove that  $\{1/n\}$  is a null sequence.

6. prove that if  $\{x_n\}$  is a null sequence and  $\{y_n\}$  is a sequence such that  $|y_n| \leq |x_n|$  for all  $n$  then  $\{y_n\}$  is a null sequence.

7. Prove that for all  $\nu \in \mathbb{Z}^+$  and all positive  $p \in \mathbb{R}$ ,  $(1+p)^\nu > \nu p$ .

8. Sequences are not mentioned in the Axioms, so we defined them as functions  $f: \mathbb{N} \rightarrow \mathbb{R}$  or  $f: \mathbb{Z}^+ \rightarrow \mathbb{R}$ , and we could define them so they start anywhere in  $\mathbb{Z}$ . We write  $f_n$  instead of  $f(n)$  to emphasize that sequences "occur in a definite order:"  $f_0$ , then  $f_1$ , then  $f_2$ , and so on. *Partial sums*, or "*summations*," are not mentioned in the

Axioms either, so we define them as follows. Given a sequence  $\{t_n\}_{n=0}^\infty$ , the symbol  $\sum_{n=0}^k t_n$  is defined by

$$(i) \sum_{n=0}^0 t_n := t_0,$$

and

$$(ii) \text{ if, for some } k \in \mathbb{N}, \sum_{n=0}^k t_n \text{ is defined, then } \sum_{n=0}^{k+1} t_n := \left( \sum_{n=0}^k t_n \right) + t_{k+1}.$$

In this way,  $\sum_{n=0}^k t_n$  is defined for all  $k \in \mathbb{N}$ . When we write  $t_0 + t_1 + \cdots + t_k$  we mean  $\sum_{n=0}^k t_n$ .

Prove that, for all real  $x$  and all positive integers  $k$ ,  $x^k - 1 = (x - 1) \sum_{n=0}^{k-1} x^n$ . This is a generalization of the familiar equation  $x^2 - 1 = (x - 1)(x + 1)$ .

9. What “should” be the value of  $\sum_{n=0}^{-1} t_n$ ? Think vacuously. . .

10. Suppose that  $x \in \mathbb{R}$ , and that for every  $\epsilon > 0$ ,  $|x| < \epsilon$ . Prove that  $x = 0$ .

### Exercises 5

Proposed for Feb 5

1. PP(13)
2. PP(14)
3. PP(19)
4. PP(20)
5. PP(21)
6. PP(22)
7. PP(23)
8. PP(25)
9. PP(27)
10. PP(31)

### Exercises 4

Proposed for Jan 31

1. PP(06)
2. PP(07)
3. PP(10)
4. PP(11)
5. PP(12)
6. Give an example of a set  $X$  and two functions  $f : X \rightarrow X$  and  $g : X \rightarrow X$  such that  $f \circ g \neq g \circ f$ .
7. (Not a “proof problem”) Let  $X$  be a set with  $n$  elements and  $Y$  a set with  $m$  elements, where  $m > n$ . How many one-to-one functions  $f : X \rightarrow Y$  are there? Why?
8. (# 7, continued) How many onto functions  $f : X \rightarrow Y$  are there? Why?
9. (# 7, continued) How many onto functions  $f : Y \rightarrow X$  are there? Why?
10. (# 7, continued) How many one-to-one functions  $f : Y \rightarrow X$  are there? Why?

### Problem 2: Due Feb 13

Suppose that  $m \in \mathbb{N}$  and  $n \in \mathbb{N}$  and  $k \in \mathbb{N}$  and  $k \neq 0$ . Prove that, if  $m = n + k$  then  $n \neq m + k$ .

### Exercises 3

Proposed for Jan 29

1. How do we define what it means for two functions to be equal?
2. We consider non-empty sets  $X$  and  $Y$ . Prove that, for all functions  $F : X \rightarrow Y$ , and all subsets  $S$  and  $T$  of  $X$ ,

$$F(S \cap T) \subseteq F(S) \cap F(T)$$

and

$$F(S \cup T) = F(S) \cup F(T).$$

3. Find examples of non-empty sets  $X$  and  $Y$  and functions  $F : X \rightarrow Y$  (one for each case) such that for some subsets  $S$  and  $T$  of  $X$ ,

$$F(S \cap T) \neq F(S) \cap F(T)$$

and

$$F(S^c) \neq F(S)^c.$$

4. In the context of # 2 and #3, what can be said about  $F(S \Delta T)$ ?

- Given a function  $F : X \rightarrow Y$ , and a set  $U \subseteq Y$ , we defined  $F^{-1}(U)$  to be the set of all  $x \in X$  such that  $F(x) \in U$ . How can we be sure that such a set exists?
- Given a function  $F : X \rightarrow Y$ ,  $F^{-1}$  need not be a function with domain  $Y$  and range  $X$ . Nevertheless, our definition of  $F^{-1}$  does determine a function. What are the domain and range of this function?
- We consider non-empty sets  $X$  and  $Y$ . Prove that, for all functions  $F : X \rightarrow Y$ , and for all subsets  $U$  and  $V$  of  $Y$ ,

$$F^{-1}(U \cap V) = F^{-1}(U) \cap F^{-1}(V),$$

$$F^{-1}(U \cup V) = F^{-1}(U) \cup F^{-1}(V),$$

$$F^{-1}(U^c) = F^{-1}(U)^c \text{ and}$$

$$F^{-1}(U \Delta V) = F^{-1}(U) \Delta F^{-1}(V).$$

- Suppose that  $X$ ,  $Y$  and  $Z$  are non-empty sets and that  $F : X \rightarrow Y$ ,  $G : Y \rightarrow Z$ . Prove that, if each function is one-to-one, so is their composite function,  $\_ \circ \_$ .
- Suppose that  $X$ ,  $Y$  and  $Z$  are non-empty sets and that  $F : X \rightarrow Y$ ,  $G : Y \rightarrow Z$ . Prove that, if each function is onto, so is their composite function,  $\_ \circ \_$ .

### Exercises 2

Proposed for Jan 24

- Suppose that  $S$  and  $T$  are subsets of a “universal” set  $X$ . We say that  $S$  is contained in  $T$  if every element of  $S$  is an element of  $T$ , and we denote this by  $S \subseteq T$ . Use set-selector notation to prove that for any subsets  $S$  and  $T$  of  $X$ , it is true that  $S \cap T \subseteq S$ .
- Suppose that  $S$  and  $T$  are subsets of a “universal” set  $X$ . What can be said about  $T$  if  $T = S \cup T$ ? Why?
- Suppose that  $S$  and  $T$  are subsets of a “universal” set  $X$ . What can be said about  $T$  if  $T = S \cap T$ ? Why?
- Suppose that  $X$  is a “universal” set. What does  $(\forall S \in 2^X)$  mean?
- Let  $X := \{a, b, c\}$  and let  $Y := \{d, e, f\}$ . How many elements does  $X \times Y$  have? How many elements does  $2^{X \times Y}$  have?
- This uses the same  $X$  and  $Y$  used in # 5. How many functions  $F : X \rightarrow Y$  are there? The notation  $Y^X$  is often used to denote the set of all functions from  $X$  to  $Y$ , for arbitrary sets  $X$  and  $Y$ .
- This uses the same  $X$  and  $Y$  used in # 5. How many functions  $F : X \rightarrow Y$  are one-to-one?
- Prove that  $U \Delta V = (U \setminus V) \cup (V \setminus U)$ .
- If  $X$  is a set that is a universe of discourse, we define the *complement* of a set  $S$  to be the set of all elements of  $X$  that are not in  $S$ . We denote the complement of  $S$  by  $S^c$ . Use set-selector notation to express  $S^c$ . Prove that  $(S^c)^c = S$ .
- Prove that  $(S \cup T)^c = S^c \cap T^c$  and that  $(S \cap T)^c = S^c \cup T^c$ . These are versions of DeMorgan’s Laws.

### Problem 1: Due Jan 30

Find “logic formulas” for each of the possible Truth Tables for  $A$  and  $B$ . Scoring will be competitive!

### Exercises 1

Proposed for Jan 22

- If  $A \Rightarrow B$  is true, and  $B$  is FALSE, what does this tell us about  $A$ ?
- Write down the Truth table for  $(\sim B \Rightarrow \sim A) \iff (A \Rightarrow B)$ .  
 $\sim B \Rightarrow \sim A$  is called the *contrapositive* of  $A \Rightarrow B$ .
- Write down the Truth table for  $(B \Rightarrow A) \iff (A \Rightarrow B)$ .  
 $B \Rightarrow A$  is called the *converse* of  $A \Rightarrow B$ .
- Suppose  $(A \wedge B) \Rightarrow C$  is FALSE. What does this tell us about the truth values of  $A$  and  $B$  and  $C$ ?
- Are  $(A \wedge B) \Rightarrow C$  and  $A \wedge (B \Rightarrow C)$  the same logically? Why?
- How many different Truth Tables for  $A$  and  $B$  are possible? Why?
- Let the truth value TRUE correspond to the number 1. Let FALSE correspond to the number 0. Find an “arithmetic” formula for each of the logical operations NOT, AND, OR, IF-THEN, IF AND ONLY IF and EXCLUSIVE OR. For example,  $\sim A$  has the formula  $1 - A$ . Thus, in your formulas, you will be replacing statements by their *numerical* truth values.

8. Suppose  $(A \vee B) \Rightarrow C$  is TRUE, and  $C$  FALSE. What does this tell us about the truth values of  $A$  and  $B$ ?
9. Three boxes sit in front of you. One of them has the label "One Dozen Roses," another the label "One Dozen Carnations." The third box bears the label "A half dozen Roses and a half dozen Carnations." You're assured that no box has the correct label! You may select one box, shut your eyes, and take a flower from it, close the box and open your eyes. Figure out how to determine which flowers are in which box. The source of this problem will be given later!