Math 3283W, Fall 2000 **Special Problems**

Special Problems are writing assignments with no rewrites. They are intended to be somewhat challenging. You should, I hope, find that they force you to ask questions! They must be well legibly and neatly written on 8.5 by 11 paper, double spaced, and in “Portrait” orientation. I expect them to be complete, in the sense that I do not, as the reader, have to supply any material you may take for granted. I also expect them to be concise! Long and rambling papers are not well written!

Special Problem 1: Due Oct. 6

Prove that for each positive integer n and for all non-negative real numbers y, the equation $x^n = y$ has a unique non-negative solution x. We denote this unique solution by $x = \sqrt[n]{y}$. Hints appear throughout the handout “On the existence of square roots” that is linked to Max Jodeit’s Web page for Math 3283W.

Special Problem 2: Due Nov. 10

For each $k \in \mathbb{N}_0$ and each $n \in \mathbb{N}_1$ let’s define

$$S_k(n) := \sum_{m=1}^{n} m^k. \text{ You know: } S_0(n) = n,\ S_1(n) = \frac{n(n+1)}{2},\ S_2(n) = \frac{n(n+1)(2n+1)}{6},\ S_3(n) = S_1(n)^2.$$

A natural question arises: How were these formulas discovered? We have a story about how Gauss discovered $S_1(n)$. In this Special problem you’ll find out how to discover the next formula, given all the preceding ones.

First Part of Special Problem 2: Prove that, if we know the formulas for $S_k(n)$ whenever k is a non-negative integer less than the positive integer K, then we can find a formula for $S_K(n)$ that is a polynomial of degree $K+1$ in n.
Second Part of Special Problem 2: In each of the known cases we see that $S_1(n)$ is a factor of $S_k(n)$. State a Theorem that asserts this and prove your Theorem.
Third Part of Special Problem 2: Find $S_4(n)$ and $S_5(n)$. Make sure your formulas are nicely simplified! Show your work and explain in words the small steps that you skip.

Special Problem 3: Due Nov. 22?

Summation by Parts and Dirichlet’s Test

Suppose we are given a series $\sum_{n=1}^{\infty} a_n b_n$.

(a) Prove that the partial sums S_n of this series can be written in terms of the partial sums $A_n := \sum_{k=1}^{n} a_k$ and the differences $b_n - b_{n+1}$ between successive terms of the sequence $\{b_n\}$, where we use the convention $A_0 = 0$ (reasonable, because the sum of no terms “ought” to be zero), as follows:

$$S_n = \sum_{k=1}^{n} A_k(b_k - b_{k+1}) + A_j b_{j+1}\Big|_{j=0}^{j=n}.$$

Find a similar formula for $S_{m+n} - S_m$ for arbitrary m and n in $\mathbb{N}_1$ (may be useful in (c)). This is *Summation by Parts.* Explain the analogy between Summation by Parts and Integration by Parts.

(b) Prove that, if $\{A_n\}$ is bounded, and if $\{b_n\}$ decreases to zero, then $\sum_{n=1}^{\infty} a_n b_n$ converges. This is *Dirichlet’s Test.*

(c) Prove, under the conditions in (b), the error estimate $|s_n - L| \le M b_{n+1}$, where $|A_n| \le M$ and $L = \sum_{n=1}^{\infty} a_n b_n$.

(d) Prove the Alternating Series Test using Dirichlet’s Test.