Math 5467, Spring 2000: Assignments

Assignments are due at the start of class on the due date.
Give credit for help received, including books and hints from me and others; mention discussions.
If a problem is difficult, please include a "narrative" that tells what you went thru to reach your results.

Assignment 6 Due May 2

1: Find the product representation $C_4 = E_1 E_2 E_3$.

2: Discuss the Discrete Cosine transform. Include an estimate for the number of multiplies needed. Hint: Consider *even* sequences of length $2N$. They have essentially the same DFT (at least some of them) as the CFT).

3: If we let

$$Tf(a,b) := |a|^{-1/2} \int \overline{\psi\left(\frac{t-b}{a}\right)} f(t)\, dt,$$

where $\int \psi(t)\, dt = 0$, and $\Gamma := 2\pi \int \frac{|\widehat{\psi}(\xi)|^2}{|\xi|} < \infty$, discuss the formula

$$f = (1/\Gamma) \int\int Tf(a,b)|a|^{-1/2} \psi\left(\frac{t-b}{a}\right) \frac{da\, db}{a^2}.$$

Assignment 5 Due Apr 5

1: Find all sets of exactly three non-zero coefficients that determine a scaling function whose integer translates form an orthonormal system.

2: Suppose we have found a scaling function $\varphi(t)$ determined by finitely many h_n. Prove that

$$\varphi(t) = \sum_n h_n \sqrt{2}\, \varphi(2t-n).$$

3: The Shannon scaling function is $\varphi(t) = \mathrm{sinc}(t) = \frac{\sin \pi t}{\pi t}$. Find $\widehat{\varphi}(\xi)$ and the coefficients $h(n)$ for its dilation equation. Find the formula for the Shannon wavelet.

4: Show that the integer translates of the Shannon scaling function form an orthonormal set.

Assignment 3 Due Feb 23

1: Find the element of V_0 that is closest to $H_{00}+H_{11}$. Is there a simple "recipe" for solving this sort of minimization problem? If so, what is it?

2: Calculate the orthogonal projection of $H_{00}+H_{11}$ onto V_{-1}. Is there a simple formula for doing this? If so, what is it?

3: Verify that the smallest closed subspace containing a given orthonormal sequence $\{v_n\}$ of vectors in a Hilbert space consists of all series of the form

$$\sum_n a_n v_n,$$

where $\{a_n\}$ is a sequence of scalars that satisfies $\sum_n |a_n|^2 < \infty$. The notes on Hilbert spaces will be useful, especially the Theorem of Riesz and Fischer.

4: Verify that, if S is a non-empty subset of a Hilbert space X that the set

$$S^{\perp} := \{x \in H : \langle x,\, s\rangle = 0 \ \text{ for all } \ s \in S\}$$

is a closed subspace of H. It is called the *annihilator* of S. A useful fact about this (not usable in solving the problem...) is that a set S has dense span in H if and only if its annihilator is $\{0\}$. "Has dense span" means that every vector in H can be approximated as accurately as desired (but still allowing a non-zero error) by a (finite!) linear combination of elements of S.

Assignment 2 Due Feb 9

1: (continuation of HW1 #4: synthesis) Examine various sums

$$\sum_{j=A}^{B} \sum_{k=-\infty}^{\infty} c_{jk} H_{jk}(t)$$

to see "how well" they approximate $f(t) = t[-1 < t < 1]$. (the c_{jk} are on the Web) I'm interested in finding out what "how well" means, from your point of view.

2: Let $\varphi(t) = B(t) = [0 < t < 1]$ be the "Box function," which is a scaling function for the Haar system. Suppose that $f(t)$ is in V_0, and suppose that f is a finite linear combination of translates of φ. That is,

$$f(t) = \sum_{j=A}^{B} c_n \varphi(t-n), \quad \text{and we define } \ c_{jk} = \int_{-\infty}^{\infty} f(t) H_{jk}(t)\, dt.$$

The integral is "really" taken over a bounded interval.

(a) Explain why $c_{jk} = 0$ if $j \geq 0$.
(b) Suppose (for this part) $\int f(t)\, dt \neq 0$. Explain why it is impossible for $c_{jk} = 0$ to be true for all j sufficiently negative. The question behind this: what is the rôle of the very-low-frequency coefficients?

3: Given j and k, which V_m does $H_{jk}(t)$ belong to?

4: Write down (and justify) the Polarization formula for inner products when the scalars are real.

Assignment 1 Due Feb 2

1: Suppose M is a 4×4 matrix with columns $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$,

which we name $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$, respectively. Given a vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$, where the x_j are complex numbers, say how to express $\mathbf{x}$ as a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$. That is, say how to find c_1, c_2, c_3, c_4 such that $\mathbf{x} = \sum_{j=1}^{4} c_j \mathbf{v}_j$. In particular, find the c_j's such that $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \sum_{j=1}^{4} c_j \mathbf{v}_j$.

2: Suppose that M is an $n \times n$ matrix of complex numbers with columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$, respectively. Suppose also that the "conjugate transpose," $\overline{M^T}$, of M is also the inverse of M. "Do" Problem 1 with this M, and compare. (Your explicit vector will now be the *column* version of $(1, 2, \ldots, n)$).

3: For $x \in \mathbb{R}$, define $H(x) := 1$ if $0 < x < 1/2$, $H(x) := -1$ if $1/2 < x < 1$, and let $H(x) := 0$ for all other x.
(i) Find j, k so that $H_{jk} = H$.
(ii) Shetch the graphs of $H_{2,1}$, $H_{-2,1}$, and $H_{3,-1}$.
(iii) Verify that the functions $H_{j,k}(x) = 2^{j/2} H(2^j x - k)$ satisfy

$$\langle H_{jk}, H_{j'k'} \rangle = \begin{cases} 1 & \text{if } j' = j \text{ and } k' = k \\ 0 & \text{if } j' \neq j \text{ or } k' \neq k. \end{cases}$$

We say the H_{jk} *form an orthonormal system of functions in* $L^2(\mathbb{R})$ (these are the Haar functions; H is the Haar wavelet).

4: In Problem 3, j refers to “scale,” and k refers to “position.” Find the Haar coefficients of the “ramp” function $r(x)$ that is equal to x for $-1 < x < 1$ and equal to zero elsewhere. That is, find

$$c_{jk} := \langle r,\, H_{jk} \rangle = \int_{-\infty}^{\infty} r(x)\overline{H_{jk}(x)}\, dx,$$

where the complex conjugation can be ignored since each H_{jk} is real-valued. I am particularly interested in how you arrange this two-parameter family of coefficients. One thought you might bear in mind is to compare your arrangement to sheet music.