Math 5615, Fall 1999 **Assignments** are due at the start of class on the given day

Assignment 9, Book Problems: Due Dec 10

Chapter 4, # 5, 7; Chapter 5, # 1, 5, 22, 24.

Special Problem 9: Due Dec 15

Return your completed Course evaluation form to the Designated Collector, and, on a separate sheet, a copy of just your Comments about the course, especially those that will be helpful to your learning the material better next semester. The Designated Collector will be asked to place the returned forms in 2 separate envelopes, take them to the office and give them to the appropriate person with the request that the Comments Only part be given to me when I turn in the grades. Your comments will be studied and hopefully influence how I conduct Math 5616.

Special Problem 8: Due Dec 13

Suppose that $f(x)$ is an increasing function defined on $[0, \infty)$, $f(0) = 0 = \lim_{x \downarrow 0} f(x)$, and suppose that, in addition, $f(x)/x$ is decreasing on $(0, \infty)$.

(a) Show that f is uniformly continuous on $[0, \infty)$.
(b) Show that f is *absolutely continuous* on $[0, T]$ for every positive real T: i.e., show that for all $\epsilon > 0$ there exists $\delta > 0$ such that if $\{I_n\}$ is a non-overlapping collection of subintervals, with endpoints $a_n < b_n$, of $[0, T]$, whose lengths add to less than δ (i.e., $\sum_n (b_n - a_n) < \delta$), then $\sum_n |f(b_n) - f(a_n)| < \epsilon$. First hint: A good example of such a function is $\sqrt{x}$.

Special Problem 7: Due Dec 8

Let us define a space into which we can embed all the $\mathbb{R}^k$ and $\mathbb{C}^k$ in a natural way: we set ℓ^2 equal to the collection of all complex sequences $x = \{x_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} |x_n|^2 < \infty$. This is usually spoken of as "little ell 2." When we want to use this to embed spaces $\mathbb{R}^k$ we can just use real-valued sequences, and we might say "*real* little ell 2." The embeddings are these: $\mathbf{v} = (v_1, \ldots, v_k) \in \mathbb{R}^k$ is transformed into the sequence whose first k terms are the v_j, $j = 1, \ldots, k$, and whose terms v_j with $j > k$ are all zero. In ℓ^2 we define the "standard orthonormal basis" to be the set of sequences e_n, $n = 1, 2, \ldots$ where e_n has all terms zero except the n-th term, which is 1. We can then express $x \in \ell^2$ as the (formal) sum $x = \sum_{n=1}^{\infty} x_n e_n$. The *norm* of $x \in \ell^2$ is defined to be $||x|| := \sqrt{\sum_{n=1}^{\infty} |x_n|^2}$, and the *inner product* of $x \in \ell^2$ and $y \in \ell^2$ is defined by $\langle x, y \rangle := \sum_{n=1}^{\infty} x_n \overline{y_n}$.
(a) State and prove the Schwarz inequality for ℓ^2;
(b) Show that $d(x, y) := ||x - y||$ is a metric on ℓ^2;
(c) Starting with $\mathbf{v}_1 := e_1$, construct *the* sequence of vectors $\{\mathbf{v}_k\}_{k=1}^{\infty}$ in ℓ^2 with these properties:

$$
\begin{array}{rl}
\text{(i)} & ||\mathbf{v}_k|| = 1 \text{ for all } k; \\
\text{(ii)} & \langle \mathbf{v}_k, e_k \rangle > 0 \text{ for all } k; \\
\text{(ii)} & \langle \mathbf{v}_k, e_j \rangle = 0 \text{ for all } j > k; \\
\text{(iv)} & \langle \mathbf{v}_j, \mathbf{v}_k \rangle = 1/2 \text{ for all } j \neq k.
\end{array}
$$

Show that this sequence of vectors satisfies $d(\mathbf{v}_j, \mathbf{v}_k) = 1$ for all $j \neq k$.
You might assign youself these problems: Show that ℓ^2 is a complete metric space; show that the property that every pair of unequal vectors are distance 1 apart can only hold in $\mathbb{R}^k$ for sets up to a certain size that depends on k, and find that size. Hint: assume that one of the vectors in the set is the vector $\mathbf{0}$.

Assignment 8, Book Problems: Due Dec 3

Chapter 3, # 12, 13, 18, 22; Chapter 4, # 4.

Assignment 7, Book Problems: Due Nov 22

Chapter 3, # 6–11.

Special Problem 6: Due Nov 15

Chapter 3, # 16(d), 16(e).

Assignment 6, Book Problems: Due Nov 10

Chapter 3, # 4, 5, 14(a), 14(b), 14(c), 16.

Special Problem 5: Due Nov 5

Suppose that $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence (for all $n \in \mathbb{Z}+$, $a_{n+1} \geq a_n$) of positive real numbers. Suppose further that the sequence $\left\{\frac{a_n}{n}\right\}_{n=1}^{\infty}$ decreases $\left(\frac{a_{n+1}}{n+1} \leq \frac{a_n}{n}\right)$ to zero $\left(\lim_{n\to\infty} \frac{a_n}{n} = 0\right)$. Prove that, for each fixed positive integer m, $a_{n+m} - a_n \to 0$ as $n \to \infty$. Construct an example showing that if the condition "$\left\{\frac{a_n}{n}\right\}_{n=1}^{\infty}$ decreases" is omitted, the conclusion may fail.

Assignment 5, Book Problems: Due Nov 1

Chapter 2, # 17, 19, 27, 28; Chapter 3, # 2, 3.

Special Problem 4: Due Oct 22

Let $\mathcal{L} := \{\emptyset, \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{R}\}$. That is, $\mathcal{L}$ consists of all unbounded open intervals that are bounded at their lower ends, together with the empty set and $\mathbb{R}$ itself.

(a) Prove that $\mathcal{L}$ is a topology for $\mathbb{R}$ (that we'll call the *lower-bound topology).*

(b) "Find" all subsets K of $\mathbb{R}$ that are compact with respect to the lower-bound topology.

Assignment 4, Book Problems: Due Oct 20

Chapter 2, # 12, 14, 15, 22 – 24, 29.

Assignment 3, Book Problems: Due Oct 8

Chapter 2, # 2, 3, 5, 7 – 9, 11.

Special Problem 3: Due Oct 4

Prove: if X is a finite set then a function $f : X \to X$ is one-to-one if and only if it is onto. Hint: use induction

Assignment 2, Book Problems: Due Sept 27

Chapter 1, # 12, 13, 16, 17.

Special Problem 2: Due Oct 1

Chapter 1, # 7.

Special Problem 1: Due Sept 22

Chapter 1, # 6.

Assignment 1, Book Problems: Due Sept 20

Chapter 1, # 2, 5, 8 – 10.