Math 5615, Fall 1999 Alternate proofs: $r^2 = 3$ has no rational solutions

The proof given in class is based on divisibility considerations. We showed that "$m^2 = 3n^2$, with m and n relatively prime," implies the contradictory "m and n have 3 as a common factor."

The alternate proofs, by Ibis Kavanaugh and by Hakan Inal, show that "$m^2 = 3n^2$, with m and n relatively prime," implies the contradictory "m and n have 2 as a common factor."

We treat the proof that $r^2 = 2$ is without rational solutions as a resource.

Case 1: $3n^2$ is even. Then n^2 is even, so m^2 is even. It follows (from the earlier proof) that m and n have 2 as a common factor.

Case 2: $3n^2$ is odd. Thus n is odd. Here the two solutions differ.

Kavanaugh: $m^2 = n^2 + 2n^2$, so $2n^2 = m^2 - n^2 = (m-n)(m+n)$.

Lemma (proof omitted): If $m^2 - n^2$ is even, so are $m+n$ and $m-n$.
(One proof is to show that $m+n$ and $m-n$ always have to have the same remainder when divided by 2.)

Writing $m+n =: 2a$ and $m-n =: 2b$ and substituting gives $2n^2 = 2a \cdot 2b$, and this implies $n^2 = 2bc$, so n is even, a contradiction.

Inal: Since $m^2 = 3n^2$, m is also odd. We write $m = 2k+1$, $n = 2\ell + 1$, and substitute:

$$\begin{aligned} m^2 = (2k+1)^2 = 4k^2 + 4k + 1 &= 3(4\ell^2 + 4\ell + 1) = 3n^2; \\ 4k^2 + 4k &= 3(4\ell^2 + 4\ell) + 2; \\ 2k^2 + 2k &= 3(2\ell^2 + 2\ell) + 1. \end{aligned}$$

The left side is even, the right side is odd and this gives us a contradiction.