

## Rearranging Absolutely Convergent Series

**Theorem:** *If a series that converges absolutely is rearranged, the new series converges to the same sum as the original one.*

You might want to skip directly to *Proof*, below! We rearrange a series by changing the order of its subscripts, making sure that we keep all the subscripts and never repeat one of them. For example, we could take the first two available odd natural numbers, then the first available even natural number, and repeat forever. This would give us a function  $p : \mathbb{N} \rightarrow \mathbb{N}$  that is one-to-one and onto, for it never repeats, and eventually picks up each natural number:

$$\begin{array}{cccccccccccccccccccccccc} k = & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & \dots \\ p(k) = & 1 & 3 & 0 & 5 & 7 & 2 & 9 & 11 & 4 & 13 & 15 & 6 & 17 & 19 & 8 & 21 & 23 & 10 & 25 & 27 & 12 & \dots \end{array}$$

This  $p$  is an example of a rearrangement or *permutation* of the natural numbers. Here are formulas for this  $p(k)$ , although the vast majority of  $p$ 's can't be given by a finite collection of formulas! Each natural number  $m$  can be written uniquely as  $m = 3q + r$ , with  $0 \leq r < 3$ . Then  $p(m)$  is given by

$$p(3j) = 4j + 1, \quad p(3j + 1) = 4j + 3, \quad \text{or} \quad p(3j + 2) = 2j,$$

depending on which form  $m$  has.

If the original series is  $\sum_{k=0}^{\infty} a_k$  the new series is  $\sum_{j=0}^{\infty} a_{p(j)}$ , whenever  $p : \mathbb{N} \rightarrow \mathbb{N}$  is one-to-one and onto. We are to prove that if  $s_n := \sum_{k=0}^n a_k \rightarrow s$  then  $t_n := \sum_{j=0}^n a_{p(j)} \rightarrow s$  for every  $p : \mathbb{N} \rightarrow \mathbb{N}$  that is one-to-one and onto.

The idea is to recognize that for large enough  $n$ ,  $t_n$  includes a partial sum  $s_{\alpha(n)}$  of the original series. That is,  $\alpha(n)$  is the largest  $\alpha$  such that  $\{0, 1, \dots, \alpha\} \subseteq \{p(0), p(1), \dots, p(n)\}$ . Then we wait until  $\alpha(n)$  is so large that  $s_{\alpha(n)} - s$  is small to assure ourselves that  $t_n - s$  is small. Let us make this precise after we look at our example again.

We have  $t_0 = a_1$  and  $t_1 = a_1 + a_3$  which do not include any partial sums of the original series. However,  $t_2 = a_1 + a_3 + a_0$  includes  $s_1$ ; we have  $t_2 = s_1 + a_3$ . Thus  $\alpha(2) = 1$  because  $\{p(0), p(1), p(2)\}$  includes  $\{0, 1\}$ . But  $\alpha(0)$  and  $\alpha(1)$  are undefined because  $0 \notin \{p(0), p(1)\}$ . We can check that  $\alpha(4) = \alpha(3) = \alpha(2) = 1$ ,  $3 = \alpha(5) = \alpha(6) = \alpha(7)$  and so on. In particular,  $\alpha(11) = 7$ . This means that  $t_{11} = s_7 + a_9 + a_{11} + a_{13} + a_{15}$ . Absolute convergence comes in now. We have

$$|t_{11} - s_7| = |a_9 + a_{11} + a_{13} + a_{15}| \leq |a_9| + |a_{11}| + |a_{13}| + |a_{15}| \leq \sum_{k=9}^{\infty} |a_k| \leq \sum_{k=8}^{\infty} |a_k|.$$

Thus we estimate the difference  $|t_n - s_{\alpha(n)}|$  by the sum of all the absolute values of the terms not included in  $s_{\alpha(n)}$ . This makes sense because all the terms in  $s_{\alpha(n)}$  are cancelled by terms in  $t_n$ .

*Proof of the Rearrangement Theorem:* There is a first  $n$  such that  $p(n) = 0$ . Call this  $n_0$ . For  $n \geq n_0$  we define  $\alpha(n)$  to be the largest  $\alpha$  such that  $\{0, 1, \dots, \alpha\} \subseteq \{p(0), p(1), \dots, p(n)\}$ . We notice that  $\alpha(n)$  increases to infinity with  $n$ , though not necessarily strictly. To see why, let a large  $M$  be given. For each  $k \in \{0, 1, \dots, M\}$  there exists (unique!)  $n_k$  such that  $p(n_k) = k$ . Thus  $\{0, 1, \dots, M\} = \{p(n_0), p(n_1), \dots, p(n_M)\}$ . If we let  $n(M) := \max\{n_0, n_1, \dots, n_M\}$  then  $\{0, 1, \dots, M\} = \{p(n_0), p(n_1), \dots, p(n_M)\} \subseteq \{p(0), p(1), \dots, p(n(M))\}$ . But for any  $n' < n(M)$ , some  $n_k > n'$  so  $k = p(n_k) \notin \{p(0), p(1), \dots, p(n')\}$ . We have shown that  $\alpha(n(M)) = M$ , and we know that if  $n \geq n(M)$ , then  $\alpha(n) \geq \alpha(n(M)) = M$ . Thus  $\alpha(n) \rightarrow \infty$  as  $n \rightarrow \infty$ .

Then for  $n \geq n_0$ ,

$$|t_n - s_{\alpha(n)}| \leq \sum_{k=\alpha(n)+1}^{\infty} |a_k|.$$

Given  $\epsilon > 0$  we choose  $K$  so large that  $\sum_{k=K+1}^{\infty} |a_k| < \epsilon/2$ . Then we choose  $N$  so large that for all  $n \geq N$ ,  $\alpha(n) \geq K$ . Now we can finish the proof. We suppose that  $n \geq N$ . Therefore  $\alpha(n) \geq K$ . Hence

$$|t_n - s| = |t_n - s_{\alpha(n)} + s_{\alpha(n)} - s| \leq |t_n - s_{\alpha(n)}| + |s_{\alpha(n)} - s| < \epsilon/2 + |s_{\alpha(n)} - s| < \epsilon.$$

The last inequality is true because  $s_{\alpha(n)} - s = -\sum_{k=\alpha(n)+1}^{\infty} a_k$ , so that, like  $|t_n - s_{\alpha(n)}|$ ,

$$|s_{\alpha(n)} - s| = \left| \sum_{k=\alpha(n)+1}^{\infty} a_k \right| \leq \sum_{k=\alpha(n)+1}^{\infty} |a_k| < \epsilon/2.$$

### The Cauchy Condensation Test

If  $\{a_n\}$  is a decreasing sequence of positive real numbers then  $\sum_{n=0}^{\infty} a_n$  converges if and only if  $\sum_{n=0}^{\infty} 2^n a_{2^n}$  converges.

When we proved that the harmonic series diverges we broke the series into blocks. This allowed us to show that a *subsequence* of partial sums diverges. Since the sequence of partial sums increases (because the terms are positive) the divergence of a subsequence implies the divergence of the whole sequence. The same idea is used in proving the Cauchy Condensation test: break the series into blocks. Then show that a subsequence of the partial sums is comparable to the partial sums of another series. This works because of the following lemma.

**Lemma** If  $\{s_n\}$  is an increasing sequence of real numbers and  $n_k$  is a strictly increasing sequence of natural numbers, let  $y_k := s_{n_k}$ . Then  $\{s_n\}$  converges if and only if  $\{y_k\}$  converges, and the two sequences have the same limit.

*Proof* of the Lemma: Suppose that  $\{s_n\}$  increases and is convergent, say  $s_n \rightarrow L$ . Then  $\{s_n\}$  is bounded above by  $L$ . Hence  $y_k = s_{n_k} \leq L$ . Thus since  $\{y_k\}$  is increasing, it converges, and  $\lim y_k \leq L$ .

Next suppose that  $\{y_k\}$  converges, say to  $M$ . Then  $y_k \leq M$  for every  $k$ . But since  $y_k = s_{n_k}$  and the  $n_k$  increase strictly and are integers,  $n_k \rightarrow \infty$ . Therefore for every  $n$ , there is a  $k$  such that  $n < n_k$ . Thus  $s_n \leq s_{n_k} = y_k \leq M$ . Hence  $\{s_n\}$  is increasing and bounded above, hence convergent, and so  $L := \lim s_n \leq M$ . Now that we know  $\{s_n\}$  is convergent, we can use the first part of the proof to show that  $M \leq L$ , so the sequences have the same limit.

*Proof* of Cauchy's Condensation Test (a shorter version follows): Let  $s_n := \sum_{k=0}^n a_k$  and  $t_n := \sum_{j=0}^n 2^j a_{2^j}$ . We will show that  $\{s_{2^n}\}$  is bounded above if and only if  $\{t_n\}$  is bounded above. The Lemma will then show that, therefore,  $\{s_n\}$  is bounded above if and only if  $\{t_n\}$  is bounded above. Since each sequence is increasing, this will show that  $\{s_n\}$  converges if and only if  $\{t_n\}$  converges. But this in turn means that  $\sum_{n=0}^{\infty} a_n$  converges if and only if  $\sum_{n=0}^{\infty} 2^n a_{2^n}$  converges.

We can use the "telescope" idea and write

$$s_{2^n} = s_0 + (s_1 - s_0) + (s_2 - s_1) + (s_4 - s_2) + \cdots + (s_{2^{j+1}} - s_{2^j}) + \cdots + (s_{2^n} - s_{2^{n-1}}) = a_0 + a_1 + \sum_{j=0}^{n-1} (s_{2^{j+1}} - s_{2^j}).$$

Then

$$s_{2^{j+1}} - s_{2^j} = \sum_{k=2^j+1}^{2^{j+1}} a_k.$$

When  $2^j < k \leq 2^{j+1}$ ,  $a_{2^j} \geq a_k \geq a_{2^{j+1}}$  because the sequence  $\{a_k\}$  is decreasing. Therefore if we replace each term in the block by the smaller one or replace each term in the block by the larger one we get

$$\sum_{k=2^j+1}^{2^{j+1}} a_{2^{j+1}} \leq \sum_{k=2^j+1}^{2^{j+1}} a_k \leq \sum_{k=2^j+1}^{2^{j+1}} a_{2^j}.$$

The block has  $2^j$  terms, so we can calculate the outer sums, and we get

$$\frac{1}{2} (2^{j+1} a_{2^{j+1}}) = 2^j a_{2^{j+1}} \leq \sum_{k=2^j+1}^{2^{j+1}} a_k \leq 2^j a_{2^j} \quad \text{and therefore}$$

$$\sum_{j=0}^{n-1} \frac{1}{2} (2^{j+1} a_{2^{j+1}}) \leq \sum_{j=0}^{n-1} (s_{2^{j+1}} - s_{2^j}) \leq \sum_{j=0}^{n-1} 2^{j+1} a_{2^{j+1}}.$$

But then 
$$a_0 + a_1 + \frac{1}{2} \sum_{j=0}^{n-1} (2^{j+1} a_{2^{j+1}}) \leq a_0 + a_1 + \sum_{j=0}^{n-1} (s_{2^{j+1}} - s_{2^j}) = s_{2^n} \leq a_0 + a_1 + t_{n-1}.$$

Since  $\sum_{j=0}^{n-1} (2^{j+1} a_{2^{j+1}}) = \sum_{j=1}^n 2^j a_{2^j} = t_n - a_1$ , the inequalities above become

$$a_0 + \frac{1}{2} a_1 + \frac{1}{2} t_n \leq s_{2^n} \leq a_0 + a_1 + t_{n-1}.$$

If  $\sum_{n=0}^{\infty} 2^n a_{2^n}$  converges, then  $\{a_0 + a_1 + t_{n-1}\}$  is bounded above. Therefore  $\{s_{2^n}\}$  is bounded above and therefore

$\sum_{n=0}^{\infty} a_n$  converges. Conversely, if  $\sum_{n=0}^{\infty} a_n$  converges then  $\{s_{2^n}\}$  is bounded above and therefore  $\{a_0 + \frac{1}{2} a_1 + t_n\}$  is

bounded above and therefore  $\sum_{n=0}^{\infty} 2^n a_{2^n}$  converges.

### Short version of the last proof

We can use the “telescope” idea and write

$$s_{2^n} = s_0 + (s_1 - s_0) + (s_2 - s_1) + (s_4 - s_2) + \cdots + (s_{2^{j+1}} - s_{2^j}) + \cdots + (s_{2^n} - s_{2^{n-1}}) = a_0 + a_1 + \sum_{j=0}^{n-1} (s_{2^{j+1}} - s_{2^j}).$$

Then

$$s_{2^{j+1}} - s_{2^j} = \sum_{k=2^j+1}^{2^{j+1}} a_k, \text{ so that } (1/2)2^{j+1} a_{2^{j+1}} = \sum_{k=2^j+1}^{2^{j+1}} a_{2^{j+1}} \leq \sum_{k=2^j+1}^{2^{j+1}} a_k \leq \sum_{k=2^j+1}^{2^{j+1}} a_{2^j} = 2^j a_{2^j}$$

because when  $2^j < k \leq 2^{j+1}$ ,  $a_{2^j} \geq a_k \geq a_{2^{j+1}}$ . Hence

$$\sum_{j=0}^{n-1} \frac{1}{2} (2^{j+1} a_{2^{j+1}}) \leq \sum_{j=0}^{n-1} (s_{2^{j+1}} - s_{2^j}) \leq \sum_{j=0}^{n-1} 2^{j+1} a_{2^{j+1}}.$$

But then 
$$a_0 + a_1 + \frac{1}{2} \sum_{j=0}^{n-1} (2^{j+1} a_{2^{j+1}}) \leq a_0 + a_1 + \sum_{j=0}^{n-1} (s_{2^{j+1}} - s_{2^j}) = s_{2^n} \leq a_0 + a_1 + t_{n-1}.$$

Since  $\sum_{j=0}^{n-1} (2^{j+1} a_{2^{j+1}}) = \sum_{j=1}^n 2^j a_{2^j} = t_n - a_1$ , the inequalities above become

$$a_0 + \frac{1}{2} a_1 + \frac{1}{2} t_n \leq s_{2^n} \leq a_0 + a_1 + t_{n-1}.$$

Thus if either series converges, so does the other, since each sequence of partial sums can be used to bound the other.