1. An elegant solution of Special Problem 2

2. A solution of A9#1

3. A question from an Office Hour

1. Verify that

$$(*) \qquad \sum_{n\in\mathbb{Z}} (-1)^n h_n h_{-n-2k-1} = 0$$

holds for *every* sequence $\{h_n\} \in \ell^2$.

Only one person (wasn't me!) solved the problem this way:
First, $(*)$ is the inner product of two square integrable sequences, so the series in $(*)$ converges absolutely. Let's call the sum p_k. Then, replacing the dummy index n (literally!) by $-n-2k-1$ gives

$$p_k = \sum_{n\in\mathbb{Z}} (-1)^n h_n h_{-n-2k-1} = \sum_{-n-2k-1\in\mathbb{Z}} (-1)^{-n-2k-1} h_{-n-2k-1} h_{-(-n-2k-1)-2k-1}.$$

Upon simplifying the indices in the last sum, we note that $(-1)^{-n-2k-1} = -(-1)^n$ and the h's become the original ones, but in the opposite order, so that $p_k = -p_k$, which can only be true if $p_k = 0$, as desired.

2. *Suppose we want h_n satisfying $(h4)$, with $h_n = 0$ unless $0 \le n \le N-1$, with $h_0 \ne 0 \ne h_{N-1}$. Show that if $(h4)$ is true for our h_n then N has to be even. Show that if $(h4)$ is true for some $k > 0$ then $(h4)$ is true for $-k$ as well. Find the number of equations that $(h4)$ produces.* Hint: For $k>0$, verify that $(h4)$ becomes $\sum_{n=2k}^{N-1} h_n \overline{h_{n-2k}} = 0$.

The equation $(h4)$ is: for $k \ne 0$, $\sum_n h_n \overline{h_{n-2k}} = 0$. In terms of convolutions and some sequence transformations we have used $(h4)$ says that $(\bar{R}h * h)_{2k} = 0$ if $k \ne 0$. Here is a general formula for this particular convolution product: given any sequence c such that $c * \bar{R}c$ makes sense, we double-conjugate, replace n by $m-n$, replace n by $-n$ then "recognize:"

$$(c * \bar{R}c)_m = \sum_n c_{m-n}\overline{c_{-n}} = \overline{\sum_n \overline{c_{m-n}} c_{-n}} = \overline{\sum_n \overline{c_n} c_{-m+n}} = \overline{\sum_n \overline{c_{-n}} c_{-m-n}} = \overline{(c * \bar{R}c)_{-m}} = [\bar{R}(c * \bar{R}c)]_m$$

so that we can write $c * \bar{R}c = \bar{R}(c * \bar{R}c)$. This means that the conjugate reflection of $c * \bar{R}c$ is the same as $c * \bar{R}c$. The formula is true for function convolution too. **Note:** The commutativity of convolution was used above!

The previous paragraph allows us to see that if $(\bar{R}h * h)_{2k} = \sum_n h_n \overline{h_{n-2k}} = 0$ then $\overline{(\bar{R}h * h)_{-2k}} = 0$. But then $(\bar{R}h * h)_{-2k} = 0$, so $(h4)$ holds for $-2k$ if it holds for $2k$. Hence the only equations in $(h4)$ that we need to worry about are those with $k > 0$.

If our equation holds for some $k > 0$ and we have $h_n = 0$ unless $0 \le n \le N-1$, and $h_0 h_{N-1} \ne 0$ then

$$\sum_n h_n \overline{h_{n-2k}} = \sum_{n=0}^{N-1} h_n \overline{h_{n-2k}} = \sum_{n=2k}^{N-1} h_n \overline{h_{n-2k}} \quad \text{because} \quad \overline{h_{n-2k}} = 0 \quad \text{if} \quad n < 2k.$$

If we could have an odd N with all our requirements satisfied, we could let $2k = N-1$. This would give

$$0 = \sum_n h_n \overline{h_{n-2k}} = \sum_{n=2k}^{N-1} h_n \overline{h_{n-2k}} = \sum_{n=N-1}^{N-1} h_n \overline{h_{n-2k}} = h_{N-1}\overline{h_{N-1-2k}} = h_{N-1}\overline{h_{N-1-(N-1)}} = h_{N-1}\overline{h_0},$$

and this contradicts "$h_0 h_{N-1} \ne 0$." We thus get $2 \le 2k \le N-2$, hence $(N-2)/2$ equations in $(h4)$.

3. The question asked came from a definition and two equations in a book shown to me, in a part about "biorthogonal systems."

The definition: $\langle x, y\rangle := \sum_n \bar{x}_n y_n$. It's the reverse of what we do, but this is not a problem.

The two equations (as well as I can remember):

$$x = \sum_n \left\langle \widetilde{f}_n, x\right\rangle f_n \qquad \text{and} \qquad x = \sum_n \langle x, f_n\rangle \widetilde{f}_n.$$

The question: what does this mean? I'm not sure, but here's an attempt at an explanation.

A quick and dirty answer: the second equation does not make sense because the inner products on the right conjugate the coordinates of x and the x on the left does not have conjugated coordinates.

Nevertheless, there is sense lurking here; the author wants to distinguish between the x's in the two equations. The x in the first equation belongs with the f_n's and the other x belongs with the $\widetilde{f}_n$'s. Thus the author wants the x's to occupy the same positions in the unner products. This can be accomplished by by defining a related but different inner product: $(x, y) := \langle y, x\rangle$. This one is defined in the way we have been using: $(x, y) = \sum_n x_n \bar{y}_n$.

Then the second equation can be written $x = \sum_n (x, f_n)\widetilde{f}_n$.

Now the question of meaning. If we had f_n instead of $\widetilde{f}_n$, the equations would look like orthonormal expansions.

The author wants to suggest the *idea* of an orthonormal expansion. To relate all this to what you may be used to, let's consider the case of real-scalar spaces first. Then the definition of $\langle x, y\rangle$ will have no complex conjugation, and we can ignore the second definition for now: both original equations make sense.

I claim that the two equations are variations on equations from linear algebra that arise when we work with invertible matrices. Let's let $\widetilde{F}$ be an $N\times N$ matrix $(\widetilde{f}_{ij})$ and let x be the $N\times 1$ column vector with components (coordinates) x_i. Then row n of $\widetilde{F}$ is the $1\times N$ matrix $(\widetilde{f}_{n1}\ \widetilde{f}_{n2}\ \cdots\ \widetilde{f}_{nN})$ and the n-th component of the $N\times 1$ column vector $\widetilde{F}x$ is $\sum_{j=1}^N \widetilde{f}_{nj}x_j = \left\langle \widetilde{f}_n, x\right\rangle$ according to the first definition. For shortness let's write $y_n := \left\langle \widetilde{f}_n, x\right\rangle$. Let's next think of f_n as *column* n of a matrix F. We know (I hope) that column n of an $N\times N$ matrix can be found using matrix operations by $f_n = Fe_n$, where e_n has all but one of its components being zero, with the non-zero one being the n-th component, and having the value one. The are often called *standard-basis vectors.*

Then the equation $x = \sum_n \left\langle \widetilde{f}_n, x\right\rangle f_n$ becomes

$$x = \sum_n \left\langle \widetilde{f}_n, x\right\rangle f_n = \sum_n y_n F e_n = F\sum_n y_n e_n = Fy = F\widetilde{F}x, \quad \text{which just means} \quad F\widetilde{F} = I.$$

Notice that we thought of $\widetilde{f}_n$ as a *row* vector and we thought of f_n as a *column* vector. Therefore we could have written $\left\langle \widetilde{f}_n, x\right\rangle$ as the matrix product $\widetilde{f}_n x$ that has size 1×1, which we then identify with the number that is its single entry. On the other hand, in the second equation we need to think of x as a *row* vector, so the author used notation in a confusing way! So lete not use x, but $\widetilde{x}$, and rewrite the second equation as $\widetilde{x} = \sum_n \langle \widetilde{x}, f_n\rangle \widetilde{f}_n$. The inner product here can once again be written as a matrix product, so that $\langle \widetilde{x}, f_n\rangle = \widetilde{x}f_n = \widetilde{x}Fe_n$ and the second equation becomes (now using: *row* n of $\widetilde{F}$ is $e_n^T\widetilde{F}$)

$$\widetilde{x} = \sum_n \langle \widetilde{x}, f_n\rangle \widetilde{f}_n = \sum_n \langle \widetilde{x}, f_n\rangle e_n^T\widetilde{F} = \sum_n \widetilde{x}Fe_ne_n^T\widetilde{F} = \widetilde{x}F\left(\sum_n e_ne_n^T\right)\widetilde{F} = \widetilde{x}FI\widetilde{F} = \widetilde{x}F\widetilde{F},$$

and this says (again) that $F\widetilde{F} = I$.

All this was done assuming we were working in a finite-dimensional real-scalar vector space. Withouit doing details, I hope you see that if we had been doing all this with complex scalars then we'd get the equations

$$F\overline{\widetilde{F}} = I \quad \text{and} \quad \overline{F}\widetilde{F} = I, \quad \text{which are really the same equation, because } I \text{ only has real entries.}$$

Since we are in the finite-dimensional case, $\overline{F}$ is the inverse of $\widetilde{F}$ and therefore $\overline{\widetilde{F}}F = I$. This means that

$$e_i^T \overline{\widetilde{F}} F e_j = \sum_{k=1}^{N} \overline{\widetilde{f}_{ik}} f_{kj} = \left\langle \widetilde{f}_i,\, f_j \right\rangle = \delta_{ij}.$$

When we turn to the case of vectors with infinitely many components, e.g. sequences, there is reason for caution. First, we need $\sum_n |x_n|^2 < \infty$ and $\sum_n |y_n|^2 < \infty$ (x and y are "square-summable"). We need the matrix F to transform square-summable rows into square-summable rows. We need the matrix $\widetilde{F}$ to transform square-summable columns into square-summable columns.

A sufficient condition for that is (for F) that there exist a positive number M such that for all n, $\sum_j |f_{jn}| \le M$. There is a similar, "transposed," version for $\widetilde{F}$. This is known as "Schur's Lemma." Finally, if we want the rows of $\widetilde{F}$ and the columns of F to be "biorthonormal," we need for F to be invertible, that is, F has to have a "two-sided" inverse. That makes the equation $\overline{\widetilde{F}}F = I$ true. Example: the matrix $\widetilde{F}$ that transforms the square-summable sequence $\{x_1, x_2, \ldots, x_n, \ldots\}$ into the square-summable sequence $\{0, x_1, x_2, \ldots, x_{n-1}, \ldots\}$ (the right-shift) has a left inverse matrix F (the left-shift) but $\widetilde{F}$ is *not* invertible.

We do get *something* in the infinite-dimensional case: if $\widetilde{F}$ has left inverse F, then the *columns* of $\widetilde{F}$ and the *rows* of F are "biorthonormal." That's what the equation $F\overline{\widetilde{F}} = I$ says.

If we want to think of our equations as equations about *functions* we have too many possibilities to say much in general. So we concentrate on the analogs of what we know about wavelets. Suppose, just for an example, that we want to treat the functions $r(t-n)$, where $r(t)$ is the "roof" function, as tho they were a scaling function and its translates. The roof function is one when $x = 1/2$, and decreases to the right and left with slopes ± 1 until its graph reaches the x-axis and is zero elsewhere (formula: $r(t) = \max\{0, 1 - |t - 1/2|\}$). It's sort of a "spread-out" Box function. For now, let's just work with a φ that is real-valued has compact support. Let's also look for our function $\widetilde{\varphi}$ to be a series in integer translates of φ: $\widetilde{\varphi}(t) = \sum_k c_k\, \varphi(t-k)$, and let's look for real coefficients c_k. The condition $\langle \varphi(s-m), \widetilde{\varphi}(s-n) \rangle_{ds} = \delta_{mn}$ becomes

$$\langle \varphi(s-m), \widetilde{\varphi}(s-n) \rangle_{ds} = \int \varphi(s-m)\widetilde{\varphi}(s-n)\, ds = \int \varphi(s-(m-n))\widetilde{\varphi}(s)\, ds = R\varphi * \widetilde{\varphi}(m-n) = 0$$

if $m - n \neq 0$. On substuting in $\widetilde{\varphi}(s) = \sum_k c_k \varphi(s-k)$ and replacing $m-n$ by ℓ we get

$$R\varphi * \widetilde{\varphi}(\ell) = \int \varphi(s-\ell) \sum_k c_k \varphi(s-k)\, ds = \int \varphi(s-(\ell-k)) \sum_k c_k \varphi(s)\, ds = \sum_k c_k R\varphi * \varphi(\ell-k).$$

Since $R\varphi * \varphi(t)$ is real, it is an even function and it has compact support. Thus the sequence $\{R\varphi * \varphi(n)\}$ is an even sequence with at most a finite number of non-zero terms. Moreover, $R\varphi * \varphi(0) = \int \varphi(s)^2\, ds > 0$. For shortening, let's write $A_n := R\varphi * \varphi(n)$. This leads to our wanting a sequence $\{c_k\}$ of real numbers such that $(A * c)_k = \delta_{k0}$. Acting on similarity with Fourier transforms we can multiply by $e^{in\xi}$ and add. After using A8#4 in the sequence context we get that we want $A(\xi)c(\xi) = 1$, which is what happens to the convolution equation $A * c = \delta$.

We require of our φ that $A(\xi)$ is never zero. Then we have to find the Fourier coefficients of $c(\xi) = 1/A(\xi)$. That is, $c_k = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-ik\xi}}{A(\xi)} d\xi$. We hope the c_k are "nice." However, there *must* be infinitely many that are non-zero! This is because our convolution equation uses *all* the k instead of just the $2k$.

The trick now is to convert our formula for c_k into a contour integral and use the residue theorem to evaluate it. In the special case of the roof function, $A(\xi) = 1 + \frac{e^{i\xi} + e^{-i\xi}}{4}$, the coefficients can be found explicitly and they *are* nice. The function $\widetilde{r}(t)$ obtained this way is piecewise linear and continuous, but does not have compact support.