Rudin's induction, paraphrased: Fix $y_1 \in \overline{\Lambda(V_1)}$. Assume $n \geq 1$ and $y_n$ has been chosen in $\overline{\Lambda(V_n)}$. We showed that each $\overline{\Lambda(V_n)}$ is a neighborhood of $0$. Hence

$$\left(y_n - \overline{\Lambda(V_{n+1})}\right) \cap \Lambda(V_n) \neq \emptyset. \tag{5}$$

This says that there exists $x_n \in V_n$ such that $\Lambda x_n \in y_n - \overline{\Lambda(V_{n+1})}$. Put $y_{n+1} = y_n - \Lambda x_n$. Then $y_{n+1} \in \overline{\Lambda(V_{n+1})}$ and the construction proceeds.

One way to describe what has been done:

Let $\eta \in \overline{\Lambda(V_n)}$. Then by (5) $\left(\eta - \overline{\Lambda(V_{n+1})}\right) \cap \Lambda(V_n) \neq \emptyset$. Hence there exists

$$x_\eta \in V_n \ \text{ such that } \ \Lambda x_\eta \in \eta - \overline{\Lambda(V_{n+1})} \ \text{ and thus } \ y_\eta := \eta - \Lambda x_\eta \in \overline{\Lambda(V_{n+1})}. \tag{6}$$

We can use this to define, for each $n \geq 1$, a function

$$h_n : \overline{\Lambda(V_n)} \to V_n \times \overline{\Lambda(V_{n+1})}. \tag{7}$$

First we well-order $V_1$. Then for each $\eta \in \overline{\Lambda(V_n)}$ we set the first coordinate of $h_n(\eta)$ equal to the *first* $x_\eta \in V_n$ that satisfies (6) and then we set the second coordinate of $h_n(\eta)$ equal to $\eta - \Lambda x_\eta \in \overline{\Lambda(V_{n+1})}$. Since the sets $V_n$ and $\overline{\Lambda(V_n)}$ are nested we can define functions $\tilde{h}_n : V_1 \times \overline{\Lambda(V_1)} \to V_1 \times \overline{\Lambda(V_1)}$ by

$$\tilde{h}_n(y) := \begin{cases} (x,\, y) & \text{if } \ y \notin \overline{\Lambda(V_n)}; \\ h_n(y) & \text{if } \ y \notin \overline{\Lambda(V_n)}. \end{cases} \tag{8}$$

Then we set $\mathcal{R}$ equal to the set of all functions $f : V_1 \times \overline{\Lambda(V_1)} \to V_1 \times \overline{\Lambda(V_1)}$ and define $H : \mathcal{R} \times \mathbb{Z}^+ \to \mathcal{R}$ by

$$H(f,\, n) := \tilde{h}_{n+1} \circ f.$$

According to the Recursion Theorem, given $f_o \in \mathcal{R}$, there exists a unique sequence $\{f_n\}$ in $\mathcal{R}$ such that $f_1 = f_o$ and $f_{n+1} = H(f_n,\, n)$. We apply this with $f_o = \tilde{h}_1$. We can express this intuitively as $f_n = \tilde{h}_n \circ \cdots \circ \tilde{h}_1$ or as

$$x_n = \pi_1(\tilde{h}_n(y_n)) = \tilde{h}_n(\tilde{h}_{n-1}(\cdots(\tilde{h}_1(y_1)\cdots)) \ \text{ and } \ y_{n+1} = \pi_2(\tilde{h}_n(y_n)) = \tilde{h}_n(\tilde{h}_{n-1}(\cdots(\tilde{h}_1(y_1)\cdots)), \ \text{ with } \ y_1 \in \overline{\Lambda(V_1)}.$$

To verify that "the construction proceeds" we need to show that each $x_n \in V_n$ and each $y_{n+1} \in \overline{\Lambda(V_{n+1})}$. This is a straightforward induction using (7) and (8).