Toward a wavelet, given our scaling function of compact support

We now want to obtain a wavelet from the scaling function we have "constructed." Thus we will still assume we have a trigonometric polynomial $m_o(\xi)$ in the rôle of low-pass filter, and that $(h0)$–$(h5)$ are true. Much of what we do won't need all those assumptions, but it's more convenient for now to have them all.

We have $V_0 \subseteq V_1$ and we want to find an orthonormal basis for $V_0^\perp \cap V_1$, namely the orthogonal complement of V_0 as a subspace of V_1.

To begin we want to know how to tell, using the Fourier transform, which f are in V_1. We know that f is in V_1 if and only if $f = \sum c_n \sqrt{2}\varphi(2t-n)$, where $\sum |c_n|^2 < \infty$. Thus

$$(S1) \qquad \widehat{f}(\xi) = \sum c_n e^{-i\xi n/2} \frac{1}{\sqrt{2}} \widehat{\varphi}(\xi/2) = \left(\sum \frac{c_n}{\sqrt{2}} e^{-i\xi n/2} \right) \widehat{\varphi}(\xi/2) =: p(\xi/2)\, \widehat{\varphi}(\xi/2),$$

where $p(\xi)$ is a 2π-periodic function in $L^2(\mathbb{T})$. Explanation: the series for f converges in the L^2 sense. As the Fourier transform is continuous on L^2, and the series converges in L^2, so does the series of Fourier transforms of the terms. Then we factor out the exponentials that arose because of translation by half-integers.

Now that we have a way to identify functions f in V_1, we ask which of them are perpendicular to V_0. We'll give the answer, then check it.

What $f \in V_1$ are orthogonal to V_0? Ans: $\widehat{f}(\xi) = \eta(\xi/2)\overline{m_o(\xi/2+\pi)}\widehat{\varphi}(\xi/2)$, $\eta \in L^2(\mathbb{T})$, $\eta(\xi+\pi) = -\eta(\xi)$.

If $f \in V_0^\perp \cap V_1$ then $\langle f, \varphi(t-n)\rangle_{dt} = 0$ for all $n \in \mathbb{Z}$. On the other hand, if $\langle f, \varphi(t-n)\rangle_{dt} = 0$ for all $n \in \mathbb{Z}$, then f is perpendicular to any linear combination of the $\varphi(t-n)$. But then f is perpendicular to the closure of the span of the $\varphi(t-n)$, which is precisely V_0. Hence all we have to do is find the members f of V_1 that are perpendicular to each $\varphi(t-n)$. As we have learned by now, we'll use the Fourier transform. By $(S1)$,

$$(S2) \qquad \langle f, \varphi(t-n)\rangle_{dt} = \int f(t)\overline{\varphi(t-n)}\, dt = \frac{1}{2\pi}\int \widehat{f}(\xi)\overline{e^{-in\xi}\widehat{\varphi}(\xi)}\, d\xi = \frac{1}{2\pi}\int p(\xi/2)\,\widehat{\varphi}(\xi/2)\overline{e^{-in\xi}\widehat{\varphi}(\xi)}\, d\xi.$$

We want the last integral in $(S2)$ to be zero for all n. By the Fourier version of the dilation equation, $\widehat{\varphi}(\xi) = m_o(\xi/2)\widehat{\varphi}(\xi/2)$ so we can insert this into the last integral in $(S2)$ and get

$$\int f(t)\overline{\varphi(t-n)}\, dt = \frac{1}{2\pi}\int e^{in\xi} p(\xi/2)\overline{m_o(\xi/2)}\, |\widehat{\varphi}(\xi/2)|^2\, d\xi.$$

We want this to be zero for every n. Once again we change variables, replacing ξ by 2ξ and periodize (see the Fifth Question part of the notes "From low-pass filter to scaling function" for details) which gives

$$\int f(t)\overline{\varphi(t-n)}\, dt = \frac{1}{\pi}\int_0^{2\pi} e^{i2n\xi} p(\xi)\overline{m_o(\xi)}\, d\xi. \text{ Which } p(\xi) \text{ make this zero for all } n?$$

When we used this before, we ended with the exponential $e^{in\xi}$ instead of $e^{i2n\xi}$, which meant we wanted the rest of the integrand back then to be a constant. We would like to use the same idea here, so we need to get rid of the factor 2 in the exponent. This is done next; we'll split $[0, 2\pi)$ into its two halves, and make a change of variables that doubles each interval (and moves the second one back to start at 0).

$$\begin{aligned}
\frac{1}{\pi}\int_0^{2\pi} e^{i2n\xi} p(\xi)\overline{m_o(\xi)}\, d\xi &= \frac{1}{\pi}\int_0^{\pi} e^{i2n\xi} p(\xi)\overline{m_o(\xi)}\, d\xi + \frac{1}{\pi}\int_\pi^{2\pi} e^{i2n\xi} p(\xi)\overline{m_o(\xi)}\, d\xi \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{in\xi} p(\xi/2)\overline{m_o(\xi/2)}\, d\xi + \frac{1}{\pi}\int_0^{\pi} e^{i2n\xi} p(\xi+\pi)\overline{m_o(\xi+\pi)}\, d\xi \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{in\xi} p(\xi/2)\overline{m_o(\xi/2)}\, d\xi + \frac{1}{2\pi}\int_0^{2\pi} e^{in\xi} p(\xi/2+\pi)\overline{m_o(\xi/2+\pi)}\, d\xi \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{in\xi} \left(p(\xi/2)\overline{m_o(\xi/2)} + p(\xi/2+\pi)\overline{m_o(\xi/2+\pi)} \right) d\xi.
\end{aligned}$$

Putting this back into $(S2)$ gives us

$$\int f(t)\overline{\varphi(t-n)}\,dt = \frac{1}{2\pi}\int_0^{2\pi} e^{in\xi}\left(p(\xi/2)\overline{m_o(\xi/2)} + p(\xi/2+\pi)\overline{m_o(\xi/2+\pi)}\right)\,d\xi,$$

which is *now* zero for *every* n if and only if

$$p(\xi/2)\overline{m_o(\xi/2)} + p(\xi/2+\pi)\overline{m_o(\xi/2+\pi)} = 0 \text{ a.e.},$$

or, simplifying by getting rid of the divisions by 2, we have found that

$(S3)$ If $f \in V_1$, then $f \perp V_0 \iff p(\xi)\overline{m_o(\xi)} + p(\xi+\pi)\overline{m_o(\xi+\pi)} = 0$ a.e., where $\widehat{f}(\xi) = p(\xi/2)\widehat{\varphi}(\xi/2)$.

The equation $p(\xi)\overline{m_o(\xi)} + p(\xi+\pi)\overline{m_o(\xi+\pi)} = 0$ has the *form* $\langle w, z\rangle = w_1\overline{z_1} + w_2\overline{z_2} = 0$, where $z := \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ is a non-zero vector in $\mathbb{C}^2$, $w := \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ is also in $\mathbb{C}^2$ and $\langle w, z\rangle$ denotes their inner product in $\mathbb{C}^2$. In other words we want $w \perp z$ in $\mathbb{C}^2$. It is an exercise in algebra to show that we must have $w = \zeta\begin{pmatrix} -\overline{z_2} \\ \overline{z_1} \end{pmatrix}$, where ζ is a (complex) constant of proportionality. To return to the matter at hand, $(S3)$, we have an equation at each point ξ, so we need to have a *function* $\zeta(\xi)$ such that

$$\begin{pmatrix} p(\xi) \\ p(\xi+\pi) \end{pmatrix} = \zeta(\xi)\begin{pmatrix} -\overline{m_o(\xi+\pi)} \\ \overline{m_o(\xi)} \end{pmatrix}.$$

This is really two equations:

$$p(\xi) = -\,\zeta(\xi)\overline{m_o(\xi+\pi)}$$

and

$$p(\xi+\pi) = \zeta(\xi)\overline{m_o(\xi)}.$$

Since these equations are to hold for all ξ, we can replace ξ by $\xi+\pi$ in both equations, and then take advantage of 2π-periodicity. When we do so, we get

$$p(\xi+\pi) = -\,\zeta(\xi+\pi)\overline{m_o(\xi)}$$

and

$$p(\xi) = \zeta(\xi+\pi)\overline{m_o(\xi+\pi)}.$$

Therefore (using the second equation before adding π and the first one after)

$$\zeta(\xi)\overline{m_o(\xi)} = p(\xi+\pi) = -\zeta(\xi+\pi)\overline{m_o(\xi)}.$$

In our special case, $m_o(\xi) \neq 0$ at all but a finite number of points, so

$$\zeta(\xi+\pi) = -\zeta(\xi) \text{ a.e.}$$

Therefore,

$$p(\xi) = -\zeta(\xi)\overline{m_o(\xi+\pi)},$$

where $\zeta(\xi)$ is 2π-periodic and has the property that $\zeta(\xi+\pi) = -\zeta(\xi)$ a.e. Moreover, in our special case $m_o(\xi) \neq 0$ if $|\xi| \leq \pi/2$, so $|m_o(\xi)| \geq A$ for some positive A if $|\xi| \leq \pi/2$, so $\zeta(\xi)$ is square-integrable on $[-\pi/2, \pi/2]$. But then, since $p(\xi) \in L^2(\mathbb{T})$ and $\zeta(\xi+\pi) = -\zeta(\xi)$ a.e., it is true that $\zeta(\xi) \in L^2(\mathbb{T})$. Thus, with $\eta(\xi) = -\zeta(\xi)$, so that now $\eta \in L^2(\mathbb{T})$, we have

$(S4)$ $p(\xi) = \eta(\xi)\overline{m_o(\xi+\pi)}$, where $\eta(\xi) \in L^2(\mathbb{T})$ and $\eta(\xi+\pi) = -\eta(\xi)$ a.e.

This completes the proof of our answer to the question: **What functions** $f \in V_1$ **are orthonormal to** V_0?

We need a better answer. The meaning of $\eta(\xi+\pi) = -\eta(\xi)$ a.e. will appear on comparing the two series in

$$\eta(\xi+\pi) = \sum \eta_n e^{in(\xi+\pi)} = -\eta(\xi) = -\sum \eta_n e^{in\xi},$$

so that

$$\sum (-1)^n \eta_n e^{in\xi} = -\sum \eta_n e^{in\xi} \quad \text{a.e.},$$

which means, by the uniqueness of Fourier coefficients, that $(-1)^n \eta_n = -\eta_n$ for all n. Thus $\eta_{2k} = 0$ for all k and we can now give an alternate answer to the original question. For now

$$\eta(\xi) = \sum_k \eta_{2k+1} e^{i(2k+1)\xi} = e^{i\xi} \sum_k \eta_{2k+1} e^{i2k\xi} =: e^{i\xi} H(2\xi), \quad \text{where} \quad H(\xi) \in L^2(\mathbb{T})$$

so that $H(\xi) = \sum_k \eta_{2k+1} e^{ik\xi} =: \sum_k H_k e^{ik\xi}$. I.e., $H_k := \eta_{2k+1}$, all k. Thus $\eta(\xi) = e^{i\xi} H(2\xi)$.

Therefore $f \in V_1 \cap V_0^{\perp}$ if and only if

$$(S5) \qquad \widehat{f}(\xi) = \eta(\xi/2)\overline{m_o(\xi/2+\pi)}\widehat{\varphi}(\xi/2) = H(\xi) e^{i\xi/2}\overline{m_o(\xi/2+\pi)}\widehat{\varphi}(\xi/2), \quad \text{where} \quad H \in L^2(\mathbb{T}).$$

A wavelet for φ**, defined via the Fourier transform**

When we stare at it awhile it will dawn on us that $(S5)$ is a formula that stands for an L^2 series expressing $f(t)$ in terms of the integer translates of the function whose Fourier transform is $e^{i\xi/2}\overline{m_o(\xi/2+\pi)}\widehat{\varphi}(\xi/2)$. To check this, let's *define* a function $\psi(t)$ by saying that its Fourier transform is

$$(S6\text{: wavelet-hat}) \qquad \widehat{\psi}(\xi) := e^{i\xi/2}\overline{m_o(\xi/2+\pi)}\widehat{\varphi}(\xi/2).$$

We can make this resemble the Fourier Scaling equation,

$$\widehat{\varphi}(\xi) = m_o(\xi/2)\widehat{\varphi}(\xi/2),$$

By *defining*

$$(S7\text{: hi-pass filter?}) \qquad m_1(\xi) = e^{i\xi}\overline{m_o(\xi+\pi)},$$

and then we'll have $\widehat{\psi}(\xi) = m_1(\xi/2)\widehat{\varphi}(\xi/2)$.

Let us return to $(S5)$ now and write

$$\widehat{f}(\xi) = H(\xi)e^{i\xi/2}\overline{m_o(\xi/2+\pi)}\widehat{\varphi}(\xi/2) = H(\xi)\widehat{\psi}(\xi) = \sum_k H_k e^{ik\xi}\widehat{\psi}(\xi) = \sum_k H_k \widehat{\left[\psi(t-k)\right]}(\xi) = \widehat{\left[\sum_k H_k \psi(t-k)\right]}(\xi).$$

Therefore $f \in V_1 \cap V_0^{\perp}$ if and only if

$$f(t) = \sum_k H_k \psi(t-k), \quad \text{where} \quad \sum_k |H_k|^2 < \infty.$$

We have shown that we can define

$$(S8) \qquad W_0 := V_1 \cap V_0^{\perp} = \overline{\text{span}\{\psi(t-k) : k \in \mathbb{Z}\}}.$$

Next, the verification that we do have a wavelt

We will call $\psi(t)$ *a wavelet* associated with the scaling function $\varphi(t)$. The term "wavelet" is applied here a little prematurely. We need to show that the family $\{\psi_{jk} : j \in \mathbb{Z},\ k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$. Before we do that, we'll show that the family $\{\psi_{0k} : k \in \mathbb{Z}\} = \{\psi(t-k) : k \in \mathbb{Z}\}$ is orthonormal. We'll develop some formulas of practical importance, then show easily that the whole family $\{\psi_{jk} : j \in \mathbb{Z},\ k \in \mathbb{Z}\}$ is orthonormal. Finally we will show that $\{\psi_{jk} : j \in \mathbb{Z},\ k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$ by showing that the only function in $L^2(\mathbb{R})$ that is orthogonal to the family is the zero function.

This "wavelet" defined in terms of the scaling function

We know that $\psi(t) \in V_1$. Our goal right now is to find the coefficients ψ_n in the L^2 series

$$\psi(t) = \sum \psi_n \sqrt{2}\varphi(2t-n).$$

We can use the same idea that led to $(S6)$. We know by $(S6)$ that

$$(S9)\qquad \begin{aligned}\widehat{\psi}(\xi) =& e^{i\xi/2}\overline{m_o(\xi/2+\pi)}\widehat{\varphi}(\xi/2)\\ =& e^{i\xi/2}\overline{\sum_n \frac{h_n}{\sqrt{2}} e^{-in(\xi/2+\pi)}}\widehat{\varphi}(\xi/2)\\ =& e^{i\xi/2}\sum_n \overline{\frac{h_n}{\sqrt{2}}} e^{in\xi/2}e^{in\pi}\widehat{\varphi}(\xi/2)\\ =& \sum_n \overline{\frac{(-1)^n h_n}{\sqrt{2}}} e^{i(n+1)\xi/2}\widehat{\varphi}(\xi/2)\\ =& \sum_n \overline{\frac{-(-1)^n h_{n-1}}{\sqrt{2}}} e^{in\xi/2}\widehat{\varphi}(\xi/2)\\ =& \sum_n \overline{\frac{-(-1)^n h_{-n-1}}{\sqrt{2}}} e^{-in\xi/2}\widehat{\varphi}(\xi/2) = m_1(\xi/2)\widehat{\varphi}(\xi/2).\end{aligned}$$

We're not quite done; so far, what we have is a formula for $m_1(\xi)$:

$$m_1(\xi) = \sum_n \overline{\frac{-(-1)^n h_{-n-1}}{\sqrt{2}}} e^{-in\xi} =: \sum_n \frac{h_n^{(1)}}{\sqrt{2}} e^{-in\xi},$$

where $h_n^{(1)} = -(-1)^n \overline{h_{-n-1}}$. To continue with $(S9)$. we have

$$\begin{aligned}\widehat{\psi}(\xi) =& \sum_n \overline{\frac{-(-1)^n h_{-n-1}}{\sqrt{2}}} e^{-in\xi/2}\widehat{\varphi}(\xi/2)\\ =& \sum_n h_n^{(1)} e^{-in\xi/2}\frac{1}{\sqrt{2}}\widehat{\varphi}(\xi/2)\\ =& \sum_n h_n^{(1)} \widehat{\left[\sqrt{2}\varphi(2t-n)\right]}(\xi)\\ =& \widehat{\left[\sum_n h_n^{(1)}\sqrt{2}\varphi(2t-n)\right]}(\xi),\end{aligned}$$

so that we have

$$(S10)\qquad \psi(t) = \sum_n h_n^{(1)}\sqrt{2}\varphi(2t-n), \quad \text{where } h_n^{(1)} = -(-1)^n\overline{h_{-n-1}} = \left\langle \psi(t),\, \sqrt{2}\varphi(2t-n)\right\rangle_{dt}.$$

The support of ψ in terms of that of φ

Suppose that $\varphi(t) = 0$ unless $M_1 \le t \le M_2$, where M_1 and M_2 are integers such that $h_n = 0$ unless $M_1 \le n \le M_2$. Then since $h_n^{(1)} = -(-1)^n\overline{h_{-n-1}}$, in $(S10)$ we have to have $M_1 \le -n-1 \le M_2$, so $-M_2-1 \le n \le -M_1-1$.

Then when $n = -M_2 - 1$, we have $\varphi(2t - n) = \varphi(2t + M_2 + 1) = 0$ unless $M_1 \le 2t + M_2 + 1 \le M_2$, or $(M_1 - M_2 - 1)/2 \le t$. When n is as large as possible, $n = -M_1 - 1$, a similar calculation yields $t \le (M_2 - M_1 - 1)/2$, so *this* wavelet $\psi(t)$ is zero unless $(M_1 - M_2 - 1)/2 \le t \le (M_2 - M_1 - 1)/2$. We notice that the length of this supporting interval is the same as that for φ, namely $M_2 - M_1$. However, its location is different unless $M_1 + M_2 = -1$. We can always translate this $\psi(t)$ by an integer, and call that translate a wavelet too. If we translate by N, i.e. replace $\psi(t)$ by $\psi(t - N)$, the new support interval will be $(M_1 - M_2 + 2N - 1)/2 \le t \le (M_2 - M_1 + 2N - 1)/2$.

Is the set $\{\psi(t-n) : n \in \mathbb{Z}\}$ an orthonormal set?

In order to show that $\{\varphi(t-n) : n \in \mathbb{Z}\}$ is an orthonormal set we proved that

$$G(\xi) := \sum_n |\widehat{\varphi}(\xi + 2\pi n)|^2 = 1 \text{ a.e.}$$

We can do the same for $\{\psi(t-n) : n \in \mathbb{Z}\}$; all we need to do is verify that

$$\sum_n |\widehat{\psi}(\xi + 2\pi n)|^2 = 1 \text{ a.e.}$$

We have

$$|\widehat{\psi}(\xi)|^2 = |m_1(\xi/2)|^2 |\widehat{\varphi}(\xi/2)|^2 = |e^{i\xi}\overline{m_o(\xi/2 + \pi)}|^2 |\widehat{\varphi}(\xi/2)|^2 = |m_o(\xi/2 + \pi)|^2 |\widehat{\varphi}(\xi/2)|^2.$$

Thus

$$\sum_n |\widehat{\psi}(\xi + 2\pi n)|^2 = \sum_n |m_o(\xi/2 + \pi n + \pi)|^2 |\widehat{\varphi}(\xi/2 + \pi n)|^2$$

We divide the n's into "evens" and "odds," as we did before. This gives us that $\sum_n |\widehat{\psi}(\xi + 2\pi n)|^2$ is equal to

$$\begin{aligned}
&\sum_n |m_o(\xi/2 + \pi 2n + \pi)|^2 |\widehat{\varphi}(\xi/2 + \pi 2n)|^2 + \sum_n |m_o(\xi/2 + \pi(2n+1) + \pi)|^2 |\widehat{\varphi}(\xi/2 + \pi(2n+1))|^2 \\
&= \sum_n |m_o(\xi/2 + \pi)|^2 |\widehat{\varphi}(\xi/2 + 2\pi n)|^2 + \sum_n |m_o(\xi/2)|^2 |\widehat{\varphi}(\xi/2 + \pi + 2\pi n)|^2 \\
&= |m_o(\xi/2 + \pi)|^2 G(\xi/2) + |m_o(\xi/2)|^2 G(\xi/2 + \pi) \\
&= |m_o(\xi/2 + \pi)|^2 + |m_o(\xi/2)|^2 = 1 \text{ a.e.}
\end{aligned}$$

We have shown that $\{\psi(t-n) : n \in \mathbb{Z}\}$ is an orthonormal set. It follows that $V_1 = W_0 \oplus V_0$.

We want to know more detail; in particular, we want a formula for expressing functions in V_1 in terms of the translates of the wavelet and the scaling function.

We need to know more than $V_1 = W_0 \oplus V_0$ by itself

One way we can be sure that $V_1 = W_0 \oplus V_0$ is to show that

$$\varphi_{10}(t) := \sqrt{2}\varphi(2t) \text{ is equal to } \sum_k \langle \varphi_{10}, \psi(s-k) \rangle_{ds} \psi(t-k) + \sum_k \langle \varphi_{10}, \varphi(s-k) \rangle_{ds} \varphi(t-k)$$

and that

$$\varphi_{11}(t) := \sqrt{2}\varphi(2t-1) \text{ is equal to } \sum_k \langle \varphi_{11}, \psi(s-k) \rangle_{ds} \psi(t-k) + \sum_k \langle \varphi_{11}, \varphi(s-k) \rangle_{ds} \varphi(t-k)$$

where as always each series is interpreted as an L^2 series. We have to show both equations, because the functions φ and ψ can only be translated by whole integer steps (in order to stay in V_0 and W_0), not the half-integer steps that work in V_1. We'll leisurely show: $\{\varphi(t-n) : n \in \mathbb{Z}\} \cup \{\psi(t-n) : n \in \mathbb{Z}\}$ is an(other) o.n. basis for V_1.

First we work on φ_{10}, and hope the same idea works for φ_{11}. We have to calculate the inner products in the series.

$$(*1)\qquad \langle \varphi_{10},\, \varphi(s-k)\rangle_{ds} = \int \sqrt{2}\varphi(2t)\overline{\varphi(s-k)}\,ds = \overline{\int \varphi(s-k)\overline{\sqrt{2}\varphi(2t)}\,ds} = \overline{\int \varphi(s)\overline{\sqrt{2}\varphi(2t+2k)}\,ds} = \overline{h_{-2k}}.$$

The last equal sign comes from our knowledge that, by definition, $\left\langle \varphi,\, \sqrt{2}\varphi(2t-n)\right\rangle_{dt} = h_n$. The same steps, using $(S10)$ (where we got the coefficients for ψ the way we did for φ), applied with $\psi(s-k)$ in place of $\varphi(s-k)$, yield

$$(*2)\qquad \langle \varphi_{10},\, \psi(s-k)\rangle_{ds} = \overline{\int \psi(s)\overline{\sqrt{2}\varphi(2t+2k)}\,ds} = \overline{h^{(1)}_{-2k}} = \overline{-(-1)^{2k}\overline{h_{2k-1}}} = -h_{2k-1}.$$

We want to show, therefore, that

$$(**)\qquad \sqrt{2}\varphi(2t) \text{ is equal to } \sum_k \overline{h_{-2k}}\,\varphi(t-k) + \sum_k -h_{2k-1}\,\psi(t-k) =: R(t)+S(t).$$

Here we finally get to use the idea of the "equality case" of the Schwarz inequality! We know $\|\sqrt{2}\varphi(2t)\| = 1$. We compute $\|R+S\|^2$ next, making use of the fact that $R \perp S$:

$$\|R+S\|^2 = \sum |h_{-2k}|^2 + \sum |h_{2k-1}|^2 = 1$$

because we're adding up the even and then the odd coefficients. The last computation needed is $\langle \varphi_{10},\, R+S\rangle$. Here's the computation for $\langle \varphi_{10},\, R\rangle$, using $(*1)$:

$$\langle \varphi_{10},\, R\rangle = \int \varphi_{10}(t)\overline{\sum_k \overline{h_{-2k}}\,\varphi(t-k)}\,dt = \overline{\int \overline{\sqrt{2}\varphi(2t)}\sum_k \overline{h_{-2k}}\,\varphi(t-k)\,dt} = \overline{\int \sum_k \overline{h_{-2k}}\,\varphi(t)\overline{\sqrt{2}\varphi(2t+2k)}\,dt},$$

where the last integral used a replacement of t by $t+k$ and some repositioning.

As we saw in $(*1)$, $\int \varphi(t)\overline{\sqrt{2}\varphi(2t+2k)}\,dt = h_{-2k}$, so

$$\langle \varphi_{10},\, R\rangle = \overline{\int \sum_k \overline{h_{-2k}}\,\varphi(t)\overline{\sqrt{2}\varphi(2t+2k)}\,dt} = \overline{\sum_k |h_{2k}|^2} = \sum_k |h_{2k}|^2.$$

The same steps, using $(*2)$, give $\langle \varphi_{10},\, S\rangle = \sum_k |h_{2k-1}|^2$. Therefore

$$\langle \varphi_{10},\, R+S\rangle = \sum |h_{-2k}|^2 + \sum |h_{2k-1}|^2 = 1 = \|\sqrt{2}\varphi(2t)\|\,\|R+S\| = \|\varphi_{10}\|\,\|R+S\|.$$

Thus

$$\|\varphi_{10} - (R+S)\|^2 = \|\varphi_{10}\|^2 - 2\mathrm{Re}\langle \varphi_{10},\, R+S\rangle + \|R+S\|^2 = 1-2+1 = 0.$$

Hence $\varphi_{10} = R+S \in V_0 \oplus W_0 \subseteq V_1$. When we repeat the argument with φ_{11} in place of φ_{10}, we get

$$(*3)\qquad \langle \varphi_{11},\, \varphi(s-k)\rangle_{ds} = \overline{\int \varphi(s)\overline{\sqrt{2}\varphi(2t+2k-1)}\,ds} = \overline{h_{-(2k-1)}}.$$

The same steps, using $(S10)$, applied with $\psi(s-k)$ in place of $\varphi(s-k)$, yield

$$(*4)\qquad \langle \varphi_{11},\, \psi(s-k)\rangle_{ds} = \overline{\int \psi(s)\overline{\sqrt{2}\varphi(2t+2k-1)}\,ds} = \overline{h^{(1)}_{-(2k-1)}} = \overline{-(-1)^{2k-1}\overline{h_{(2k-1)-1}}} = h_{2k-2}.$$

Hence $\varphi_{11} \in V_0 \oplus W_0 \subseteq V_1$ as well. This means that the translate of φ_{10} to the right by $1/2$, namely φ_{11}, is also in $V_0 + W_0$. Since we can translate either of these two functions φ_{10} and φ_{11} to the right or left by any integer (we translate the whole series in each case) we can see that every translate of φ_{10} by integer/2 is in $V_0 + W_0$. But then $\text{span}\{\sqrt{2}\varphi(2t-k) : k \in \mathbb{Z}\} \subseteq V_0 \oplus W_0$. Since $V_0 \oplus W_0$ is closed,

$$V_1 = \overline{\text{span}\{\sqrt{2}\varphi(2t-k) : k \in \mathbb{Z}\}} \subseteq V_0 \oplus W_0 \subseteq V_1,$$

which is what we wanted to prove. But we obtained more by going through these details.

"Reversing" the Scaling Equation!

We are used to seeing $\phi(t)$ expressed in terms of the translates of $\sqrt{2}\varphi(2t)$. We can use what we have just done to express $\sqrt{2}\varphi(2t)$ and its *half-integer* translates in terms of the *integer* translates of $\varphi(t)$ and of $\psi(t)$. So far we know that

$$\varphi_{10}(t) = \sum_k \overline{h_{-2k}}\varphi(t-k) + \sum_k -h_{2k-1}\psi(t-k) \quad \text{and} \quad \varphi_{11}(t) = \sum_k \overline{h_{-2k+1}}\varphi(t-k) + \sum_k h_{2k-2}\psi(t-k).$$

This gives us, for each $n \in \mathbb{Z}$, two equations:

$$\varphi_{1,2n}(t) = \sqrt{2}\varphi(2t-2n) = \sqrt{2}\varphi(2(t-n)) = \varphi_{10}(t-n) = \sum_k \overline{h_{-2k}}\varphi(t-n-k) + \sum_k -h_{2k-1}\psi(t-n-k)$$

and

$$\varphi_{1,2n+1}(t) = \sqrt{2}\varphi(2t-2n-1) = \sqrt{2}\varphi(2(t-n)-1) = \varphi_{11}(t-n) = \sum_k \overline{h_{-2k+1}}\varphi(t-n-k) + \sum_k h_{2k-2}\psi(t-n-k)$$

for the half-integer translates of φ_{10}. In each sum in each equation, let us replace k by $k-n$. The equations become

$$\varphi_{1,2n}(t) = \sum_k \overline{h_{-2k+2n}}\varphi(t-k) + \sum_k -h_{2k-2n-1}\psi(t-k); \; \varphi_{1,2n+1}(t) = \sum_k \overline{h_{-2k+2n+1}}\varphi(t-k) + \sum_k h_{2k-2n-2}\psi(t-k).$$

Now suppose we have $f(t) \in V_1$. Then $f(t) = \sum_m \langle f, \varphi_{1m}\rangle \varphi_{1m}(t) =: \sum_m c_m \varphi_{1m}(t)$. We can write this in terms of φ_{0k} and ψ_{0k}:

$$\begin{aligned}
P_1 f(t) = f(t) &= \sum_n c_n \varphi_{1n}(t) \\
&= \sum_n c_{2n}\varphi_{1,2n}(t) + \sum_n c_{2n+1}\varphi_{1,2n+1}(t) \\
&= \sum_n c_{2n} \sum_k \overline{h_{-2k+2n}}\varphi(t-k) + \sum_n c_{2n} \sum_k -h_{2k-2n-2}\psi(t-k) \\
&+ \sum_n c_{2n+1} \sum_k \overline{h_{-2k+2n+1}}\varphi(t-k) + \sum_n c_{2n+1} \sum_k h_{2k-2n-2}\psi(t-k) \\
(S10.5) \qquad &= \sum_k \left(\sum_n c_{2n}\overline{h_{-2k+2n}} + c_{2n+1}\overline{h_{-2k+2n+1}} \right) \varphi(t-k) \\
&+ \sum_k \left(-\sum_n c_{2n}h_{2k-2n-2} + c_{2n+1}h_{2k-2n-2} \right) \psi(t-k) \\
&= \sum_k \left(\sum_n c_n \overline{h_{-2k+n}} \right) \varphi(t-k) \\
&+ \sum_k \left(\sum_n (-c_{2n} + c_{2n+1})h_{2k-2n-2} \right) \psi(t-k).
\end{aligned}$$

A digression to develop some notation The coefficients on the φ and ψ translates in $(S10.5)$ have messy expressions. Mathematicians like to make up notation to express things – and this propensity is not limited to mathematicians!

Five transformations of sequences

If we transform a sequence $\{x_n\}$ into the sequence $\{x_{-n}\}$ we define the new sequence Rx by $(Rx)_n := x_{-n}$. It's the *reflection* of $\{x_n\}$ about the origin. It is also useful to define the reflected-and-conjugated sequence , $\bar{R}x$, by $(\bar{R}x)_n = \overline{x_{-n}}$.

We have, several times, divided a sequence into two sequences, one that uses the even-subscripted terms only, and one that uses only the odd-subscripted terms. Let's define two new transformations, $\mathcal{E}x$ and $\mathcal{O}x$ by $(\mathcal{E}x)_n := x_{2n}$ and $(\mathcal{O}x)_n := x_{2n+1}$. These transformations delete either the odd-subscripted terms or the odd ones and then "close up," removing the zeroes that fill the omissions. We'll also need $\mathcal{J} := \mathcal{O} - \mathcal{E}$. Thus $(\mathcal{J}x)_n = c_{2n+1} - c_{2n}$.

Convolution of sequences

By analogy with the convolution of functions, we define the *convolution* of sequences $x = \{x_{-n}\}$ and $y = \{y_{-n}\}$, denoted $x * y$, by

$$(x * y)_k = \sum_n x_{k-n} y_n, \quad \text{and we check that } \ x * y = y * x \ \text{ and } \ x * (y * z) = (x * y) * z.$$

The convolutions have to make sense for these equations to be true. The first one is always true. The convolution "makes sense" if every one of the series $\sum_n x_{k-n} y_n$ converges absolutely, as k runs over all the integers. Examples of cases for which the Associative Law (the second equation) is true include: two of the three sequences make absolutely convergent series (="absolutely summable"), and the third is bounded; all three are absolutely summable, two are (absolutely) square-summable and the third is absolutely summable.

These notations give some shortening of the coefficients in the writing of $f \in V_1$ as the sum of a series in W_0 and a series in V_0. For we have $\sum_n c_n \overline{h_{-2k+n}} = (c * \bar{R}h)_{2k}$ and $\sum_n (-c_{2n} + c_{2n+1}) h_{2k-2n-2} = [[(\mathcal{O} - \mathcal{E})c] * \mathcal{E}h]_{k-1}$. This "notation" business helps us "see" something interesting. The coefficients in the φ series use *half* the terms in $c * \bar{R}h$; the coefficients in the ψ series are a "full" convolution, but the convolution uses only *half* of the h_n's, the even-subscripted ones, and it combines adjacent terms in the c sequence. This can lead to nice algorithms, with good speed. We'll return to this point later. Right now we summarize the preceding discussion by writing out the equations $(S10.5)$ using the new notation, rearranged a little:

$$(S10.6) \qquad \begin{aligned} P_1 f(t) = f(t) &= \sum_n c_n \varphi_{1n}(t) \\ &= \sum_k (\bar{R}h * c)_{2k} \varphi(t-k) + \sum_k [\mathcal{E}h * (\mathcal{O} - \mathcal{E})c]_{k-1} \psi(t-k). \end{aligned}$$

The first sum is $P_0 f$, the projection of f on V_0 and the second sum is $Q_0 f$, the projection of f on W_0.

An important application of $(S10.6)$

Let $N > 1$. If $f \in V_N$, then $f(t) = 2^{(N-1)/2} g(2^{N-1}t)$, where $g(t) \in V_1$. Moreover, $\|g\| = \|f\|$ because of the normalizing factor $2^{(N-1)/2}$. Also, if we let $d_n := \langle g, \varphi_{1n} \rangle$ then $d = \{d_n\}$ plays the rôle of c in $(S10.6)$. Thus

$$g(t) = \sum_n \langle g, \varphi_{1n} \rangle \varphi_{1n}(t) = \sum_n d_n \varphi_{1n}(t) = \sum_k (\bar{R}h * d)_{2k} \varphi(t-k) + \sum_k [\mathcal{E}h * \mathcal{J}d]_{k-1} \psi(t-k),$$

where $\mathcal{J} := \mathcal{O} - \mathcal{E}$ for short. But then

$$f(t) = 2^{(N-1)/2} g(2^{N-1}t) = \sum_k (\bar{R}h * d)_{2k} 2^{(N-1)/2} \varphi((2^{N-1}t - k) + \sum_k [\mathcal{E}h * \mathcal{J}d]_{k-1} 2^{(N-1)/2} \psi(2^{N-1}t - k).$$

A little shorter:

$$f(t) = \sum_k (\bar{R}h * d)_{2k}\, \varphi_{N-1,k}(t) + \sum_k [\mathcal{E}h * \mathcal{J}d]_{k-1}\, \psi_{N-1,k}(t).$$

Now let's look at the d_n more closely. By definition, then a change of variables, then some simplification,

$$\begin{aligned} d_n = \langle g,\, \varphi_{1n}\rangle &= \int g(t)\overline{\sqrt{2}\varphi(2t-n)}\,dt \\ &= \int g(2^{N-1}t)\overline{\sqrt{2}\varphi(2\cdot 2^{N-1}t-n)}\,2^{N-1}dt \\ &= \int 2^{(N-1)/2}g(2^{N-1}t)\overline{\sqrt{2}\cdot 2^{(N-1)/2}\varphi(2\cdot 2^{N-1}t-n)}\,dt \\ &= \int 2^{(N-1)/2}g(2^{N-1}t)\overline{2^{N/2}\varphi(2^N t-n)}\,dt \\ &= \int f(t)\overline{2^{N/2}\varphi(2^N t-n)}\,dt = \langle f,\, \varphi_{Nn}\rangle\,. \end{aligned}$$

This says that d_n is the projection coefficient of f with respect to φ_{Nn}. Thus, **if we start with the projection coefficients of $f(t)$ on V_N, exactly the same rule for splitting $g \in V_1$ into $Q_0 g + P_0 g \in W_0 \oplus V_0$ is used for splitting $f \in V_N$ into $Q_{N-1}f + P_{N-1}f \in W_{N-1} \oplus V_{N-1}$.**

We can now show that ψ is a wavelet!

We have seen that $V_1 = V_0 \oplus W_0$. Since each space V_j can be obtained from V_0 by a dilation depending on j, we can use the same dilations to define the spaces W_j and then we get $V_{j+1} = V_j \oplus W_j$ for every integer j. Then we can say that, starting with any N,

$$(S11) \qquad V_N = W_{N-1} \oplus V_{N-1} = \cdots = W_{N-1} \oplus W_{N-2} \oplus \cdots \oplus W_{N-M} \oplus V_{N-M}.$$

This is what is often wanted for applications. We can start with a signal $f(t)$ in V_N (actually, the projection of a "real" signal onto V_N) and write it as the sum of a "low-frequency" signal in V_{N-M} and a sum of signals with finer and finer detail, each one in a wavelet space of a different scale. Each of the wavelet spaces is perpendicular to the other wavelet spaces, and to V_{N-M}.

We would like to be sure that this works fully. We would like to be able to prove that the family of the ψ_{jk} forms an orthonormal basis for $L^2(\mathbb{R})$.

$(S11.4)$ **Theorem:** *For every $f \in L^2$,*

$$(S11.5) \qquad f(t) = \sum_j \sum_k \langle f,\, \psi_{jk}\rangle \psi_{jk}(t),$$

an L^2 series, where $\psi_{jk}(t) = 2^{j/2}\psi(2^j t - k)$.

To prove this we recall that all we have to do is show that the only function (i.e., "vector") in L^2 that is orthogonal to every one of the ψ_{jk} is the zero function.

Proof: We know that the union of all the spaces V_j is dense in L^2. We can build on this information by showing that if f is orthogonal to every one of the ψ_{jk} then f is orthogonal to every one of the spaces V_j, hence orthogonal to their union. We can work one j at a time for the spaces V_j, so let's work with V_N. We are given that $f \perp \psi_{jk}$ for all j and k. This means that $f \perp W_j$ for all j. Let's let P_N denote the orthonormal projection on V_N. Then by $(S11)$,

$$(S12) \qquad P_N f = \sum_{j=1}^{M} \sum_k \langle P_N f,\, \psi_{N-j,k}\rangle \psi_{N-j,k}(t) + P_{N-M} f.$$

Since $N > N-j$ in this range, we have $\langle P_N f,\, \psi_{N-j,k}\rangle = \langle f,\, P_N \psi_{N-j,k}\rangle = \langle f, \psi_{N-j,k}\rangle = 0$. Hence $P_N f = P_{N-M} f$ for all $M > 0$. Thus $P_N f \in V_j$ for all $j \le N$. But since the spaces are nested, $P_N f \in V_j$ for all $j > N$ as well. Hence $P_N f \in V_j$ for all j. We know that the intersection of all the spaces V_j is the zero subsapce of L^2. Thus $P_N f = 0$ for all N. But $P_N f = 0$ means that $f \perp V_N$. Since N was arbitrary, $f \perp V_N$ for all N, so

$f \perp \bigcup_j V_j$, hence $f = 0$. **This completes the proof that the wavelets ψ_{jk} form an orthonormal basis for** L^2. Finally, we have shown that ψ *is* a wavelet.

Remarks

The equations $(S11.5)$ and $(S12)$ carry with them the idea that $P_N f \to f$ in L^2 as $N \to +\infty$, and the idea that $P_N f \to 0$ in L^2 as $N \to -\infty$. These ideas are correct.

Introduction to applications of wavelets

Analysis of a signal

Suppose we start with a signal $f(t) \in L^2$ that has compact support. Usually we assume that $f(t)$ has as much smoothness as we need. But sometimes we have to deal with jump discontinuities. At any rate, we pick a finest scale of detail, N, and a coarsest one, $N - M$. We might sample the signal at lots of points, and make the assumption that $f(t_k) \approx 2^{N/2}\langle f, \varphi_{Nk}\rangle$, where the points t_k are pretty close to $k/2^{-N}$, and the k are chosen so that the $k/2^{-N}$ are in a supporting interval for f. The factor $2^{N/2}$ is there for normalizing – the integal of $2^{N/2}\varphi_{Nk}$ is one. If f were constant, say v, on the support of φ_{Nk}, we'd want the coefficient suitably adjusted. In other words, we think of

$$P_N f \approx f_S(t) := \sum_{k \in S} 2^{-N/2} f(t_k)\varphi_{Nk}(t)$$

where the sum is taken over those k in a finite set S that corresponds to a support interval for f. Then we use $(S12)$, and say

$$P_N f(t) = \sum_{j=1}^{M}\sum_k \langle P_N f,\, \psi_{N-j,k}\rangle \psi_{N-j,k}(t) + P_{N-M}f(t) \approx f_S(t) = \sum_{j=1}^{M}\sum_k \langle f_S,\, \psi_{N-j,k}\rangle \psi_{N-j,k}(t) + P_{N-M}f_S(t).$$

Each sum $Q_{N-j}f_S(t) := \sum_k \langle f_S,\, \psi_{N-j,k}\rangle \psi_{N-j,k}(t)$ is finite and its coefficients $\langle f_S,\, \psi_{N-j,k}\rangle$ can be computed exactly. Each f_{Sj} shows the detail that is present at the scale 2^{N-j}. Likewise, the term $P_{N-M}f_S(t)$ is a finite sum whose coefficients, $\langle f_S,\, \varphi_{N-M,k}\rangle$, can be computed exactly. It shows the coarsest-scale details.

In all of this, "computed exactly" means, "expressed in terms of the numbers h_n." The only trouble is that these numbers may not be known exactly, so we once again have to rely on approximations.

Synthesis of a signal

We may want to save an approximation of a signal. One thing that is often done is to save the largest few percent of the coefficients, along with which wavelets or scaling functions they go with. Then only those coefficients and wavelets and scaling functions that are relevant are used to reconstruct, or synthesize, the signal (approximation).

Synthesis of other kinds of functions

Although wavelets "fit" into the context of Hilbert spaces $L^2(\mathbb{R}^n)$, it has been found that many other spaces of functions can be synthesized using wavelets. Chapters 5 and 6 in the book of Hernandez and Weiss are devoted to this topic, and there are many papers on this subject of ongoing research.

Some of the kinds of functions wavelets are used to synthesize are smooth functions. Thus smooth wavelets are needed. This is where the finite sets of h_n's come in; with more work, we can arrange to have wavelets that have compact support and continuous derivatives of order up to any integer $N > 0$. The work involves (in effect) linear equations involving the h_n's that arise by demanding that more and more derivatives of $m_o(\xi)$ are zero when $\xi = \pi$ (we already require that $m_o(\pi) = 0$).

It is interesting that infinitely differentiable wavelets with compact support cannot exist!